Lecture 3: July 31

Lecturer: Naveen Garg *Scribes: Khaleeque Ansari**

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

(*Based on notes by Douglas B. West)

Last Lecture:

- Finding a matching in a General Graph (Blossom Algorithm)
- Tuttes Theorem: A graph $G = (V, E)$ has a perfect matching if and only if for every subset U of V, the subgraph induced by $V - U$ has at most $|U|$ connected components with an odd number of vertices.

3.1 Gallai Edmonds Decomposition

In a graph G, let B be the set of vertices covered by every maximum matching in G, and let $D = V(G) - B$. Further partition B by letting A be the subset consisting of vertices with at least one neighbor outside B, and let $C = B - A$. The Gallai-Edmonds Decomposition of G is the partition of $V(G)$ into the three sets A, C, D.

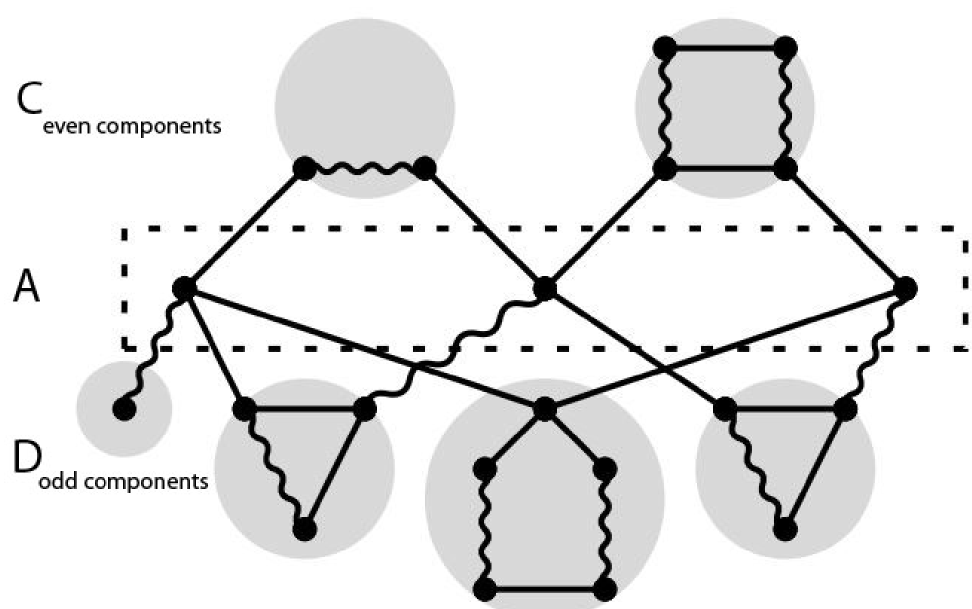

Figure 3.1: The Gallai-Edmonds Decomposition of G

Theorem 3.1 *(Gallai-Edmonds Structure Theorem) Let A,C,D be the sets in the Gallai Edmonds Decomposition of a graph G. Let G_1, . . . ,G_k be the components of G[D]. If M is a maximum matching in G, then the following properties hold.*
a) Every vertex in C is matched.
b) Every vertex in A is matched to distinct components in G[D].
c) Each component in D, G_i, is factor critical.*
d) If $\phi \neq S \subseteq A$, then $N_G(S)$ has a vertex in at least $|S|$+1 of G_1, . . . ,G_k.
(A graph is said to be factor critical if G-v is a perfect matching.)*

Claim 3.2 *Each odd component in G - A is factor critical.*

Proof: Consider the odd component G_i.

$$v \in G_i$$

We have to prove that $G_i - v$ has a perfect matching i.e. $G_i - v$ has no tutte set. Lets prove this by contradiction. Say there is a tutte set T.

$$C_o(T \cup v) \geq |T| + 1$$

Using Parity argument, odd components parity is same as that of T.

$$C_o(T \cup v) \geq |T| + 2$$

Now consider the set $A \cup T \cup v$, the number of odd components for this set is at least

$$|D| - 1 + C_o(T \cup v)$$

Hence the deficiency of this set is at least

$$|D| - |A|$$

So we have got a new set with larger cardinality for the same deficiency. Therefore there is a contradiction to fact that A was the maximal set with the largest deficiency.

Hence proved it is factor critical. ■

Claim 3.3 *Every vertex in C is matched.*

Proof: Consider the even component G_j. We have to prove that G_j has a perfect matching i.e. G_j has no Tutte set. Following the same arguments as for **Claim 3.2**

Say there is a tutte set T.

$$C_o(T) \geq |T| + 1$$

Again using Parity argument, odd components parity is same as that of T.

$$C_o(T) \geq |T| + 2$$

Now consider the set $A \cup T$, the number of odd components for this set is at least

$$|D| + C_o(T)$$

Hence the deficiency of this set is at least

$$|D| - |A| + 2$$

So we have got a new set with larger deficiency. Therefore there is a contradiction to fact that A was the set with the largest deficiency.

Hence proved there is no Tutte set. ■

Claim 3.4 *If $\phi \neq S \subseteq A$, then $N_G(S)$ has a vertex in at least $|S|+1$ of $G_1, \ . \ . \ . \ ,G_k$.*

Proof: Let T be a maximal set among the vertex sets of maximum deficiency in G. For $T \subseteq V(G)$, define an auxiliary bipartite graph $H(T)$ by contracting each component of $G - T$ to a single vertex and deleting edges within T with Y denoting the set of components of $G - T$, the graph $H(T)$ is a T, Y -bigraph having an edge ty for $t \in T$ and $y \in Y$ if and only if t has a neighbor in G in the component of $G - T$ corresponding to y. For $S \subseteq T$, all vertices of $Y - N_{H(T)}(S)$ are odd components of $G - (T - S)$. Because T is the set with maximum deficiency, we have $def(T - S) = (|Y| - |N_H(S)|) - |T - S| \leq def(T)$. Since $def(T) = |Y| - |T|$, the inequality simplifies to $|S| \neq |N_H(S)|$. Thus Halls Condition holds, and $H(T)$ has a matching that covers T. Let R (not an empty set) be a maximal subset of T for which equality holds. The crucial point is that $C = R \cup R\prime$, where $R\prime$ consists of all vertices of all components of $G - T$ in $N_{H(T)}(R)$. Since $|N_{H(T)}(R)| = |R|$, the edges of M match R into vertices of distinct components of $G[R\prime]$. We have observed that M covers the rest of $R\prime$. Since M covers T and no vertex of R or $R\prime$ has a neighbor in the other odd components of $G - T$, we conclude that $R \cup R\prime \subseteq C$. Let $D\prime = V(G) - T - R\prime$. It suffices to show that $D = D\prime$ and $A = T - R$. That is, we show that every vertex in $D\prime$ is omitted by some maximum matching and that every vertex of $T - R$ has a neighbor in $D\prime$. Let $H\prime = H(T) - (R \cup N_{H(T)}(R))$. For $S \subseteq T - R$ with$S$ nonempty, we have $|N_{H\prime}(S)| > |S|$, since otherwise R could be enlarged to include S. Therefore, deleting any vertex of $N_{H\prime}(T - R)$ from $H\prime$ leaves a subgraph of $H\prime$ satisfying Halls Condition, so $H\prime$ has a maximum matching omitting any such vertex. And also each component of $G - T$ is factor-critical, so each vertex in $D\prime$ is avoided by some maximum matching. Hence proved.

■