Lecture 2: 29th July, 2013

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2.1 Matching in General Graphs

In the case of bipartite graphs, there is an easy characterization on the existance of perfect matching in the graph. Now, we give a similar characterization for the existance of perfect matching in general graphs.

Let $S \subseteq V$ be any subset of V. Consider the graph on the vertex set $V \setminus S$. This graph, $G_{V \setminus S}$, is obtained by removing vertices in S from V and the edges associated with them. The edges in this subgraph would exist only between the connected components formed by removing vertices and the set of vertices in S. Let us denote the set of odd components in $G_{V \setminus S}$ by $C_o(S)$.

Definition 2.1 *For any $S \subseteq V$, S is called a Tutte set if $|C_o(S)| > |S|$.*

Theorem 2.2 *[Tutte's Theorem] Graph G has a perfect matching if and only if $\forall S \subseteq V$, $|C_o(S)| \leq |S|$ (G does not have a Tutte set).*

Proof: [only if] Assume that G has a perfect matching. We need to prove that G does not have a Tutte set. We prove this by contraposition. Assume that there exists a Tutte set in G. To prove, G can't have a perfect matching. For perfect matching, one vertex in each odd component has to be matched with a vertex in S. Since, the number of odd components are more than the number of vertices in S then some vertices of these odd components have to remain unmatched. Therefore, G can not have a perfect matching.

[if] Assume that G does not have a Tutte set. To prove that G has perfect matching. Again we proof this by contraposition. We assume there exists a vertex which can not be matched. We will show the existance of a Tutte set in the graph.

We provide a constructive version of the proof here. Starting from an unmatched vertex (initially all the vertices are unmatched, pick any vertex) we construct an alternating tree. The procedure of building an alternating tree is mentioned in the following. We consider all unmatched edges from an unmatched vertex and only one matched edge from each matched vertex. In the alternating tree, vertices are leveled even or odd depending on the length of the alternating path from the starting unmatched vertex to that vertex. We mark o against the vertices of odd level and mark e for even level vertices. Unlike in the case of bipartite graphs, edges between vertices in the same level are allowed for general graphs.

A typical alternating tree for an unmatched vertex in shown in Figure 2.1.

We search for augmenting paths in the alternating tree built for the starting unmatched vertex. Whenever we come across an augmenting path, we use that augmenting path to increase the size of matching by 1 and continue with the alternating tree of another unmatched vertex. We also look for a special structure in the alternating tree called *blossom*.

Definition 2.3 *A blossom is an odd cycle in which only one vertex is unmatched and the remaining vertices are matched using (matched) edges of the cycle.*

Figure 2.1: An alternating Tree

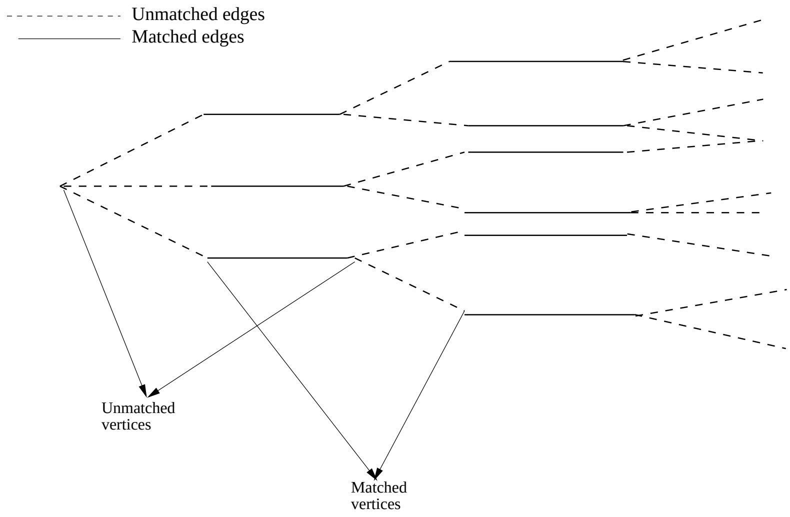
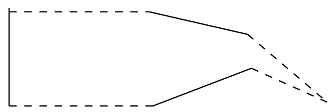


Figure 2.2 shows a blossom.

Figure 2.2: A blossom



Edmond's polynomial time algorithm (blossom algorithm) for finding perfect matchings in non-bipartite graphs works by detecting blossoms in an alternating tree for some unmatched vertex. The idea is to find blossoms in the alternating tree and shrink it (replace the blossom by an unmatched vertex). This process is repeated in the modified graph.

If there exists an unmatched edge between two vertices of an even level or if there exists a matched edge between two vertices of an odd level, then they form a blossom. Blossoms in an alternating tree are shown in Figure 2.3.

Whenever a blossom is detected in an alternating tree, the blossom is contracted and replaced by an unmatched vertex. A typical blossom contraction is shown in Figure 2.4.

The algorithm for finding perfect matching in a non-bipartite graph is roughly the following. Search for augmenting paths in the alternating trees. If a blossom is detected, shrink the blossom and continue. When an augmenting path is found, increase the size of matching and start with another unmatched vertex. We repeat this process till there are augmenting paths or blossoms in the graph.

Let's assume a blossom is detected in a graph G. Let G' be the modified graph when the blossom is contracted in G. Next, we prove a theorem establishing the equivalence between finding perfect matching in G and perfect matching in the modified graph G'.

Theorem 2.4 *There exists a perfect matching in the contracted graph (G') if and only if there exists a*

Figure 2.3: Blossoms in alternating tree

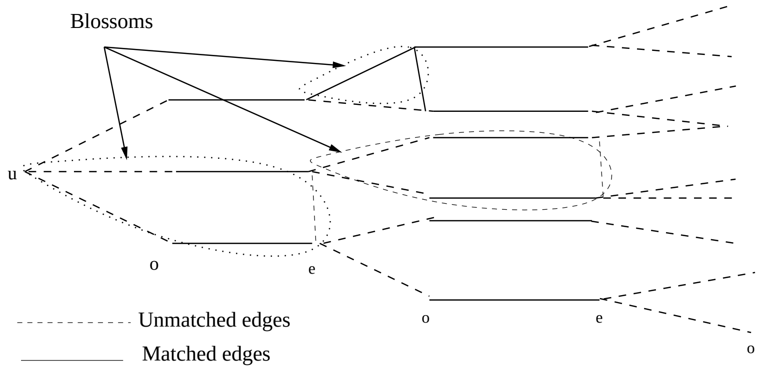
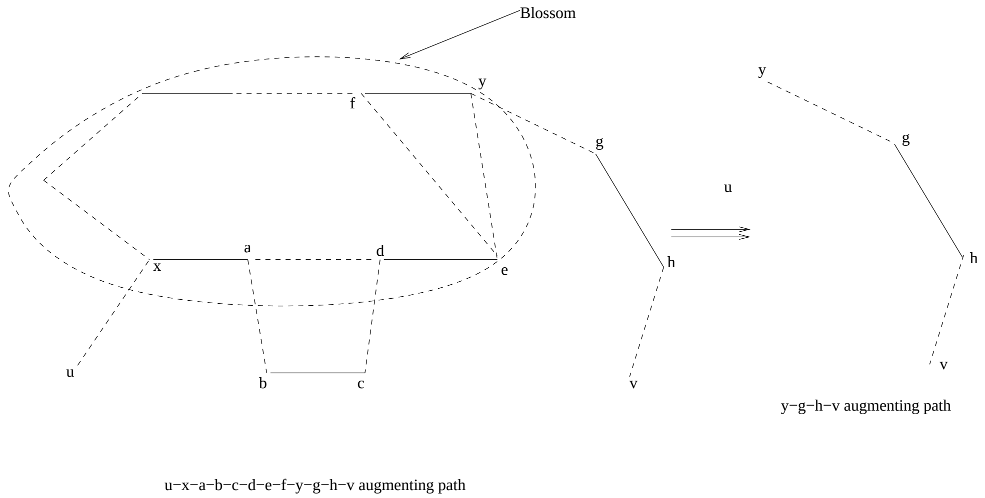


Figure 2.4: Blossom contraction



perfect matching in the original graph (G). Here G' is obtained by contracting a blossom in G.

Proof: Let M be the matching in G and M is not a perfect matching. Then, there exists an augmenting path P in G. We prove that if G has an augmenting path P then G' also has an augmenting path P'. Therefore, G' is also not a perfect matching. The other direction is proved in similar fashion.

[if] Assume G has an augmenting path. To prove that G' also has an augmenting path.

We consider two cases.

- Case 1: The augmenting path in G is disjoint with the blossom. Therefore, even when the blossom is contracted, the augmenting path will be present in G' as well.
- Case 2: The augmenting path P in G overlaps with the blossom B. We observe that at least one of the endpoints of P, let's call it s, is not in B. Let t be the first vertex where P intersects B. Then, even

after contraction of B, $(s - t)$ will be part of an augmenting path in G'. Therefore, an augmenting path exists in G'.

[only if] Assume G' has an augmenting path. To prove that the graph G also has an augmenting path.

Let us assume that G' has a matching M' and it has an augmenting path P'. Therefore, there exists some matching N' in G' of greater cardinality than M'. The augmenting path P' will cut contracted blossom B' in G' only at one point. Now, when this blossom is expanded to recover G from G', edges of the blossom can be added to generate a larger augmenting path in G. Assume that earlier matching in G was M and the new matching is N. Therefore, $|N| = |N'| + k > |M'| + k = |M|$ where $k > 1$. Therefore, an augmenting path exists in G. ■

Proof of the main theorem relies on the following reasoning. We assume that the perfect matching in the graph G does not exist. That is there exists a vertex which can not be matched in the graph. Now, assume that we build the alternating tree for that vertex and start finding blossoms and contracting it. Assume we reach a stage where the alternating tree of the vertex does not have any more blossoms and also we can not find any more augmenting paths. We can say the following things at that stage.

- There can not be any unmatched edge present between any two vertices in an even level. Had there been such an unmatched edge, it could have formed a blossom.
- There can not be any matched edges present between any two vertices in an even level. Vertices in the even level are reached via matched edges from the odd level vertices. Therefore, these vertices can not have matched edges among themselves.
- Can there be any edges (matched/unmatched) present between two vertices belonging to two different even layers? There can not be unmatched edges between vertices across two different even levels. The reason is while building the alternating tree, we considered all the unmatched edges from the vertices in the even levels. Therefore, the other ends of these edges have to be in odd level.

  There can not be any matched edges between vertices across different even levels. The vertices in the even level are matched except the starting unmatched vertex. Therefore, there can not be another matched edge with that vertex.
- There are no matched edges between any two vertices in an odd level. Had there been such a matched edge between two vertices in an odd level, they would have formed a blossom.
- Can there be any unmatched edge present between any two vertices in an odd level? They can of course be present.
- Can there be any edges (matched/unmatched) present between two vertices belonging to two different odd levels. Since, this is a matching, therefore, matched edges between such two vertices are not possible. But, unmatched edges can be present between vertices across two different odd levels.

Suppose we start an unmatched vertex and build its alternating tree. We find blossoms in the tree and shrink them. Assume that we come to a stage where there is no augmenting path or a blossom. We state the following claim.

Claim 2.5 *Consider S to be the set of vertices in the odd level of the alternating tree. We want to prove that removing $|S|$ vertices from G creates more than $|S|$ odd components.*

Proof: There are no edges (matched or unmatched) between vertices in even levels. The only edges to vertices in even level are from vertices in the odd levels. Therefore, removing these odd vertices creates components. Now, due to construction, the number of even vertices is one more than the number of odd vertices. The starting unmatched vertex may be a blossom. But, since a blossom has odd number of vertices, it will be an odd component. Also, every other vertex in any even level will be a singleton (odd) component when S is removed.

Since, the number of odd components is more than the number of vertices removed from the graph, a Tutte set exists. Therefore, perfect matching is not possible. ■

Therefore, we start with a vertex which can not be matched and show that a Tutte set exists. Therefore, the other direction is proved. ■