CSL851: Algorithmic Graph Theory **Semester I,2013**

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

12.1 Introduction to Probability

Definition 12.1 *Let (Ω, Σ, P) be a finite probability space, where Ω is a sample space which is the set of all possible outcomes, Σ is a set of events where each event is a set containing zero or more outcomes, and P is the assignment of probabilities to the events. Let X and Y be a Random Variables (RV) on Ω. Then,*

1. *Expectation of X is $E[X] = \Sigma_{w \in \Omega} X[w]P(w)$*
2. *Variance of X is $Var[X] = E[(X - E[X])^2]$*
3. *Covariance of X and Y is $Cov[X, Y] = E[(X - E[X])(Y - E[Y])]$*

Variance measures the spread of a distribution.

Proposition 12.2 *If X,Y be RVs*

1. $E[X] = \Sigma_{x \in X(\Omega)} xP(X = x)$
2. $Var[X] = E[X^2] - E[X]^2$
3. $E[X + Y] = E[X] + E[Y]$
4. $Var[X + Y] = Var[X] + Var[Y] + 2 * Cov[X, Y]$
5. $Cov[X, Y] = E[XY] - E[X]E[Y]$

Theorem 12.3 *Let X,Y be independent RVs*

1. $E[XY] = E[X]E[Y]$
2. $Cov[X, Y] = 0$
3. $Var[X + Y] = Var[X] + Var[Y]$

Covariance is a weak measure of independence. Also, $Cov[X, Y] = 0$ does not imply that X,Y are independent.

Definition 12.4 *Let $P \in [0, 1]$ and let X be an RV*

1. *X is p-bernoulli RV (or p-bernoulli distributed) if* $X : \Omega \implies \{0, 1\}$ *and*
   $P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$
2. *X is* $B(n, p)$ *binomially distributed if* $X : \Omega \implies \{0, 1, ..., n\}$ *and*
   $P(X = x) = \binom{n}{x} p^x (1 - p)^{1-x}$

Proposition 12.5 *Examples of bernoulli and binomial distributions.*

1. *p-bernoulli models the flip of a p-biased coin*
2. *Let* $X_1, ...X_n$ *be p-bernoulli RVs. Define* $X = \Sigma_{i=1}^{n} X_i$ *and suppose* $X_i s$ *are independent, then, X is B(n,p) distributed*

Theorem 12.6 *Let X be a RV.*

1. *If X is p-bernoulli, then*
   $E[X] = p$ *and* $Var[X] = p(1 - p)$
2. *If X is B(n,p), then*
   $E[X] = np$ *and* $Var[X] = np(1 - p)$

Proof:

1. Let X be p-bernoulli.
   (a)
   $$\begin{aligned} E[X] &= 1 * p + 0 * (1 - p) \\ &= p \end{aligned}$$
   (b)
   $$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= E[X] - E[X]^2 \quad // \; since\ X = X^2\ for\ p - bernoulli \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$
2. Let X be $B(n, p)$.
   (a)
   $$\begin{aligned} E[X] &= E[\Sigma_{i=1}^{n} X_i] \\ &= \Sigma_{i=1}^{n} E[X_i] \quad // \; from\ proposition\ 1.12\ (4) \\ &= np \end{aligned}$$
   (b)
   $$\begin{aligned} Var[X] &= Var[\Sigma_{i=1}^{n} X_i] \quad // \; from\ proposition\ 1.15\ (2) \\ &= \Sigma_{i=1}^{n} Var[X_i] \quad // \; since\ X_i\ are\ independent \\ &= np(1 - p) \end{aligned}$$

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12.2 Concentration Inequalities

Theorem 12.7 ***Markov's inequality***
Let X be an RV and $X \geq 0$, then,

1. *For any $\lambda > 0$, $P(X > \lambda) \leq \frac{E[X]}{\lambda}$*
2. *For a monotone increasing function $g : \mathbb{R} \implies \mathbb{R}_+$, we have $P(X \geq \lambda) \leq \frac{E[g(x)]}{g(\lambda)}$*

Theorem 12.8 ***Chebychev's inequality***
Let X be an RV. For any $\lambda > 0$, we have, $P(|X - E[X]| \geq \lambda) \leq \frac{Var[X]}{\lambda^2}$

Theorem 12.9 ***Hoeffding's inequality***
Let $X_1, ..., X_n$ be independent RVs with $a_k \leq X_k \leq b_k$ for all $k = 1, 2, ..., n$. Let $X = \Sigma_{k=1}^{n} X_k$ and let $c_k = b_k - a_k$, then for any $\lambda > 0$,

1. $P(X - E[X] \geq \lambda) \leq e^{-\frac{2\lambda^2}{\Sigma_{k=1}^{n} c_k^2}}$
2. $P(X - E[X] \leq -\lambda) \leq e^{-\frac{2\lambda^2}{\Sigma_{k=1}^{n} c_k^2}}$
3. $P(|X - E[X]| \geq \lambda) \leq 2e^{-\frac{2\lambda^2}{\Sigma_{k=1}^{n} c_k^2}}$

Corollary 12.10 ***Chernoff bound***
Let $X_1, ..., X_n$ be independent RVs with $P(X_k = 1) = p_k$ and $P(X_k = 0) = 1 - p_k$, $p_k \in [0, 1]$, and let $X = \Sigma_{k=1}^{n} X_k$. Then for any $\lambda > 0$, we have, $P(|X - E[X]|) \geq \lambda) \leq 2e^{-\frac{2\lambda^2}{n}}$

Corollary 12.11 *Let $X_1, ..., X_n$ be independent $\{-1, +1\}$ valued RVs with $P(X_k = +1) = P(X_k = -1) = 0.5$. Then for any $\lambda > 0$, we have, $P(|X| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2n}}$*

Proof: Idea of proof of Hoeffding's theorem:

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\begin{aligned}
&P(X - E[X] \geq \lambda) \\
&\qquad = P(e^{t(X - E[X])} \geq e^{t\lambda}) \\
&\qquad \leq e^{-t\lambda} E[e^{t(X - E[X])}] \quad // \; Markov's\ inequality \\
&\qquad \leq e^{-t\lambda} e^{E[t(X - E[X])]} \quad // \; t\ is\ chosen\ to\ obtain\ the\ tightest\ bound
\end{aligned}
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