CSL851: Algorithmic Graph Theory **Semester I 2013-2014**

Lecture 7: November 11

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7.1 Planar Graphs

Definition 7.1 *a planar graph is a graph that can be drawn on a plane in such a way that it's edges intersect only at their end points. In other words, it can be drawn on the plane with no edges cross each other.*

Figure 7.1:

In this lecture our main motivation is to solve NP-Hard optimization problems such as independent set, vertex cover on planar graphs. In planar graphs, the problems maximum independent set and minimum vertex cover remains NP-complete to find exactly but may be approximated to within any ratio $c < 1$ in polynomial time. In this lecture we state planar seperator theorem, using this we solve the maximum independent set probelm on planar graphs.

7.2 Planar seperator theorem

Theorem 7.2 *In any planar graph we can partition the vertex set into three sets A,B and C, where* $|B| \leq 4\sqrt{n}, |A|, |C| \leq 2n/3$ *such that no vertex in A is adjecant to vertices in C.*

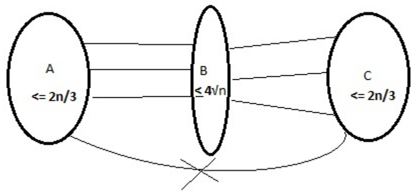

Figure 7.2:

Proof: A brief introduction to the proof of planar seperator theorem is given below. complete proof is discussed in the next class. ■

Theorem 7.3 *In any planar graph $G(V, E)$ with $|V| \geq 3$, $|E| \leq 3|V|$*

Proof: let v,e and f is the number of vertices,edges and faces in the given planar graph.

Claim 7.4 $3f \leq 2e$

take any planar graph and for every face in the planar graph count the number of edges adjecant to it and take the sum. let k be the total sum.

as we know every edge in the planar graph can share atmost two faces. so $k \leq 2e$, and every face in the planar graph can have atleast 3 edges so $3f \leq k$.

$$3f \leq k \leq 2e$$
$$3f \leq 2e$$

Euler's theorem states that in any planar connected graph $v - e + f = 2$.

$$\Rightarrow 2 \leq n - e + 2e/3$$
$$2 \leq n - e/3$$
$$e \leq 3n - 6$$
$$|E| \leq 3|V|$$

■

7.3 Maximum independent set

Theorem 7.5 *Any planar graph can be colored with 4-colors.*

Corollary 7.6 *Any planar graph has an independent set of size $\geq n/4$.*

We use divide and conquer in combanitation with planar seperator theorem to find good approximate solution for max independent set in planar graphs.

Procedure: let G(V,E) be the given connected planar graph. using planar seperator theorem partition the vertices into three set A,B and C . remove setB vertices from the graph and apply planar seperator on the subgraphs formed by sets A and C. Repeat this procedure until each seperated subset have size O(logn). find out the maximum independent set in each subpart and return the union.

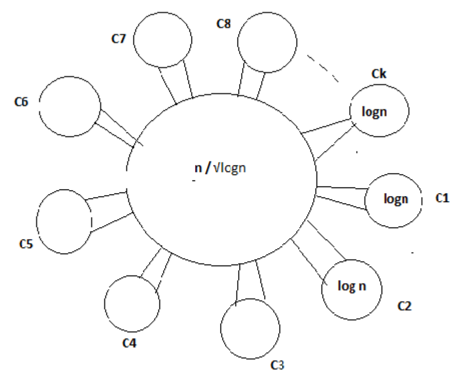

Figure 7.3:

what is the total number of the vertices left out in the above partitioning preocedure ?
it is O$(n/\sqrt{logn})$. the proof of this is given in the next class.

Analysis: Let $C_1, C_2, ..., C_k$ are the subparts formed in the above procedure. $A_1, A_2, ...A_k$ are the maximum independent sets found in each subpart $C_1, C_2, ..., C_k$ respectively. Let O is the optimal independent set.

$$\text{let } O_i = O \cap C_i$$
$$|O_i| \leq |A_i|$$
$$\Rightarrow \sum_i |A_i| \geq \sum_i |O_i| \geq |O| - \frac{n}{\sqrt{logn}}$$

from corollary 4.6

$$\sum_i |A_i| \geq \frac{n}{4} - \frac{n}{\sqrt{logn}}$$

if $\frac{n}{\sqrt{logn}} \leq \frac{\varepsilon n}{4} \Rightarrow n \geq 2^{\frac{16}{\varepsilon^2}}$ then

$$|O| - \frac{n}{\sqrt{logn}} \geq |O| - \frac{\varepsilon n}{4}$$
$$\geq |O| - \varepsilon|O| = (1-\varepsilon)|O|.$$

if $n < 2^{\frac{16}{\varepsilon^2}}$ then use brute force technique.

7.4 Triangulated planar graph

Definition 7.7 *a planar graph in which all faces (including the outer one) are bounded by three edges is called triangulted planar graph.*

Proof: Idea of proof of planar seperator theorem

Build a triangulated planar graph from the given graph by adding new edges to G in such a way that every face in the new graph $G^{'}$ is bounded by only three edges.

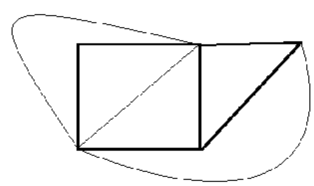

Figure 7.4: Triangulated planar graph

let B is outermost cycle in the graph $G^{'}$. A and C are sets of vertices inside and outside the cycle B respectively. then $|B| = 3, |A| > 2n/3$ and $|C| = 0$. let $k = 2\sqrt{n}$

repeat the following steps until we get the sets A,B and C with suitable sizes.

- **if** $|B| < 2k$:

  Find an edge (u,v) on the outer cycle B such that it lies in a triangle with the third vertex(x) inside B. let P be the path between u and v through vertex x. $B \backslash (u, v) \cup P$ forms the new outer cycle. doing this increases the size of B by one and decreases the size of A by one where as the size of C doesn't change.

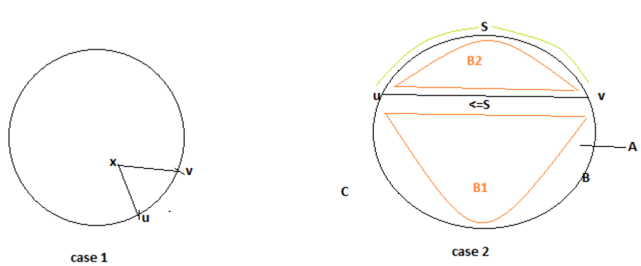

Figure 7.5:

- **if** $|B| \leq 2k$:

  Find two vertices u and v on B with d(u,v) minimum such that $d(u, v) \leq c(u, v)$ (where c(u,v) , d(u,v) be the number of edges in the shortest path between u and v on cycle B and inside the cycle B respectively). let P be the path between u and v with d(u,v) edges. now B,$B_1$and B_2 be the three cycles of $B \cup P$ where $|A(B_1)| \geq |A(B_2)|$. where A(I) = $A \cap$ {all the vertices inside cycle I}
  now B_1 will be new B, all the vertices inside it forms the set A and the vertices outside it forms the set C.

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