Lecture 5: September 24

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

5.1 Threshold for connectedness in Random Graphs

Theorem 5.1 *Let $\alpha = \alpha(n)$ be a function with $\alpha(n) \to \infty$ as $n \to \infty$. Then $S(n) := \frac{\ln(n)-\alpha(n)}{n}$ is a lower threshold function and $t(n) := \frac{\ln(n)+\alpha(n)}{n}$ is an upper threshold function for connectivity of G(n,p).*

Proof:

1. Lower Threshold Function: If $p \leq s$ then by Theorem 4.3 G(n,p) has isolated vertices approximately almost surely. Therefore, G(n,p) graph are not connected.
2. Upper Threshold Function: We assume $\alpha(n) \leq \ln(n)$ for all n. For each $k \leq n$, let X_k be the random variable which counts the number of connected components of size k.

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Let $Y = \sum_{k \leq n} X_k$ and let $X := \sum_{k \leq \lfloor \frac{n}{2} \rfloor} X_k$. Note here Y counts all the connected components in G(n,p) whereas X counts all the connected components of size at most $\lfloor \frac{n}{2} \rfloor$.
We thus have $Y \geq 1$ implies $X \geq 1$ because there is some smaller connected component inside the bigger component.
Therefore(Using Markov inequality), $P(Y \geq 1) \leq P(X \geq 1) \leq \frac{E(X)}{1} = E(X)$

Claim 5.2 *$E(X) \to 0$ as $n \to \infty$*

Proof: Consider $S \subseteq V$,
P(S forms a maximal connected component in G(n,p)) $\leq$ P(no vertex in S is connected to S^c)
P(no vertex in S is connected to S^c) $= (1-p)^{|S|(n-|S|)} \leq e^{-p|S|(n-|S|)} \leq e^{-t|S|(n-|S|)}$
Note the last inquality holds if $p \geq t$ (and we can this Equation 1).
Now we are going to analyze the expectation of each $X_k \forall k \leq \lfloor \frac{n}{2} \rfloor$

$$E(X_k) \leq e^{-tk(n-k)} \binom{n}{k}$$

Now we need to give a good estimation of the Binomial Coefficient. Using Stirling's formula :

$$E(X_k) \leq \left(\frac{ne}{k}\right)^k e^{-tk(n-k)}$$

Next inserting the formula for t we obtain :

$$E(X_k) \leq e^{-\alpha(n)} \left(\frac{e^{1-(1-\frac{1}{n})\alpha(n)} + \frac{k}{n}(\ln(n) + \alpha(n))}{k} \right)^k = e^{-\alpha(n)}(B_k)^k$$

We can the above Equation 2 and now turn our attention to B_k.
Case 1: When $k \leq \lfloor n^{\frac{3}{4}} \rfloor$ then :

$$\frac{k}{n}(\ln(n) + \alpha(n)) \leq \frac{1}{n^{\frac{1}{4}}}(\ln(n) + \alpha(n)) \leq \frac{2\ln(n)}{n^{\frac{1}{4}}} \to 0$$

The last approximation can be realized using the Taylor Series expansion. Henceforth, the numerator in (2) is at most a constant say c ¿ 0. If c ¡ k, then with $\theta = \frac{c}{c+1}$ we have $B_k \leq \frac{c}{k} \leq \theta < 1$ for any $k \geq c + 1$.
So, $\sum_{k=c+1}^{\lfloor n^{\frac{3}{4}} \rfloor}(B_k)^k \leq \sum_{k=c+1}^{\lfloor n^{\frac{3}{4}} \rfloor}\theta^k \leq c_1$ for some constant $c_1 > 0$
Case 2: When $\lfloor n^{\frac{3}{4}} \rfloor \leq k \leq \lfloor \frac{n}{2} \rfloor$ then:

$$B_k \leq e^{1-\frac{1}{4}\ln(n)-(\frac{1}{2}-\frac{1}{k})\alpha(n)}$$

Since $k \geq 2$, $\frac{1}{2} - \frac{1}{k} \geq 0$ so the exponent tends to infinity. So $B_k \leq \theta$ for some $\theta < 1$
Therefore, $\sum_{k=\lfloor n^{\frac{3}{4}} \rfloor}^{\frac{n}{2}}(B_k)^k < c_2$ for some constant $c_2 > 0$.
Thus $E(X) = \sum_{k=1}^{\frac{n}{2}} E(X_k) \leq e^{-\alpha(n)}(c_1 + c_2 + c_3) \to 0$ as $n \to \infty$.

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