Lecture 1: October 13

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

1.1 Finding Vertex Cover of a graph

1.1.1 Algorithm

Let Δ be the upperbound on d(v), v$\in$ V(G). As done in finding Independent set, we will find S$\subset$V(G), $\|S\|$ be o($\frac{n}{\sqrt{\log n}}$) such that, in induced sub graph formed by removing S from V(G), size of individual components is O($logn$). Let X_i denote the vertex cover for component G_i.G_0 is the central component.
Output of the algorithm is $\Sigma_i X_i$ + S.
Note: Using bruteforce technique minimum vertex cover for a component of size k can be found out in O(2^k). In our case k being $logn$ makes it linear in n. There are maximum $\frac{n}{\log n}$ such components and hence the algorithm terminates polynomial time.

Claim 1.1 *Let O be the optimal Vertex cover. Let T be the vertex cover given by this algorithm. Let $O_i \subset G_i, O_i \subset O$.*
*Then size(T)$\leq$(ϵ+1)*size(O) as long as $\epsilon \geq \Delta * c/(\sqrt{logn})$*

Proof: Let O_i be vertices of O in G_i. $\|O_i\| \geq \|G_i\|$
$\Rightarrow \Sigma_i \|O_i\| \geq \Sigma_i \|X_i\|$
$\Rightarrow \|G_0\| + \Sigma_i \|O_i\| \geq \|G_0\| + \Sigma_i \|X_i\|$
Now $\|G_0\| = cn/\sqrt{logn} \leq \epsilon \frac{n}{\Delta}$ from the assumption
and together with $\frac{n}{\Delta} \leq \|O\|$, we get { since size of any vertex cover$\geq \frac{m}{\Delta} \geq \frac{n}{\Delta}$}
$\Rightarrow (\epsilon + 1)\|O\| \geq G_0 + \Sigma_i \|X_i\|$ $\{Since \Sigma_i \|O_i\| \leq \|O\|\}$
$\Rightarrow (\epsilon + 1)\|O\| \geq \|X\|$
Hence vertex cover given by the algorithm is near optimum.

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1.2 Planar Separator Theorem

Theorem 1.2 *In any planar graph G(V,E), we can partition V into sets A,B,C with $\|B\| \leq 4\sqrt{n}; \|A\|, \|C\| \leq 2n/3$such that induced graph on A is not connected with induced graph on C by any edge in E.*

Proof: For a given V, we will prove for maximum edged planar graph. It is then trivial to show for planar graphs with less edges as the same sets A, B,C will satisfy the theorem constraints.

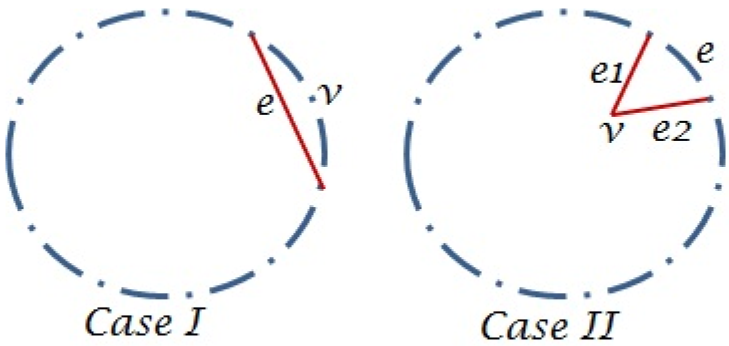

Figure 1.1: Case 1 of the proof

Maximum edged planar graph is necessarily a triangulated graph (Length of every face is 3). Otherwise we can always add an edge inside a face with more than 3 vertices thereby violating maximality of the previous graph. Here is a constructive proof of the theorem.

1.2.1 Algorithm

Invariants:

- B will form a cycle. In the beginning B will comprise of the vertices forming the outermost face of the graph.
- k=$2\lfloor\sqrt{n}\rfloor$
- $\|B\| \leq 2k$
- $\|C\| \leq \frac{2n}{3}$

Stop when $\|A\| \leq \frac{2n}{3}$.
Iterative Steps:

1. $\|B\| < 2k \rightarrow$ {Refer fig 1.1}
    - Case I: Remove v from B. B still forms a cycle through e.
    - Case II: Add v to B. B still forms a cycle with e1 and e2 edges as can be seen evidently.
2. $\|B\| = 2k \rightarrow$ { Refer Fig 1.2} Find v1, v2 $\in$ B such that path between them exists using only the edges inside the circle and let p1 be such shortest path. It should also hold that length of p1 is smaller than equal to length of the shortest path between v1 and v2 using edges from the circle. We will prove later that such a pair v1 and v2 exists. if V(G1)$\geq$ V(G2) then we will make A=G1. Otherwise A=G2. Vertices forming Boundary of A will be B and the remaining vertices will be C.

1.2.1.1 Correctness

If we are in case 1, and in sub-case I, Size of B decreases by 1 and size of C increase by 1. Before this step $\|C\| < \frac{2n}{3}$ and so invariants are not violated. If we are in sub-case II, Size of A decreases by 1. Size of B increases by 1 and C's size remains constant. Since $\|B\| < 2k$ before this step, no invariants are violated. If we are in case 2,

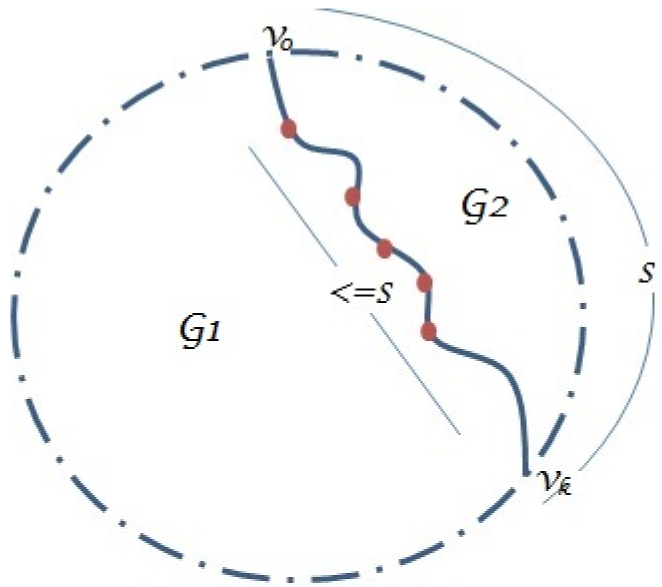

Figure 1.2: Case 2 of the proof

- Since size of path(v1,v2) inside the old B is less than equal to size of path(v1,v2) using edges of the old circle and $\|B\| = 2k$, after this step $\|B\| \leq 2k$.
- Since old $\|A\| > \frac{2k}{3}$ and since we are taking bigger of the two divisions of the graph enclosed by B, after this step $\|A\| > \frac{k}{3}$ and hence $\|C\| < \frac{2n}{3}$.

So we have proved that in any step in the algorithm, invariants are not violated.What remains to be proved is that we will get the mentioned conditions in those cases.

- Case 1:
- Case 2: Let $v_1, v_2, ..., v_k, .., v_{2k}$ be the ordering of vertices in C, C being the cycle formed from B. We define two sets:
  $V1 = \{v_1, v_2, ..v_{k+1}\} V2 = \{v_{k+1}, ..v_2 k\}$

  Claim 1.3 *There are k+1 vertex disjoint paths from V1 to V2.*

  Using the claim we find that only way to satisfy the claim as well as maintain the planarity of the graph is by having path from v_2 to v_{2k}, v_3 to v_{2k-1} and so on. See {fig 1.3 } Now a path from v_{1+j} to v_{2k-j+2} will contain atleast 2j edges and hence 2j-1 internal vertices. A simple counting of the number of vertices present in those k+1 vertex disjoint graphs gives lower bound $k^2/2$.Putting k in terms of n, we find this greater than n which is therfore a contradiction.

Claim 1.4 *There are k+1 vertex disjoint paths from V1 to V2.*

Proof: (See fig 1.4)We add 2 vertices x1 and x2 and add from them an edge to each member of V1 and V2 respectively. Assuming claim to be false, then from the vertex version of Menger's theorem, we will get a set of vertices P=$\{p_1, p_2, ..p_j\}$ where j≤k which will disconnect x1 from x2. Due to triangulation property, it can be shown that P will form a path. Also v_1, v_{i+1} will be present in P since they form a path of length 0. So there exist a path of length less than equal to k. If no other vertex of C is present in P then this is a contradiction as we have assumed there being no paths between v_1, v_{k+1}

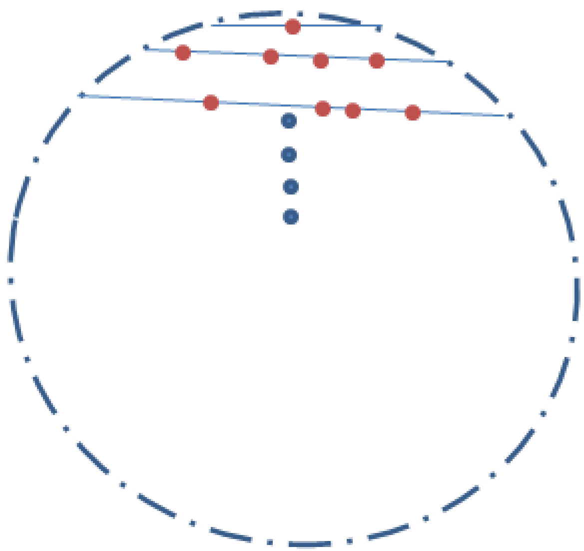

Figure 1.3: k+1 disjoint paths

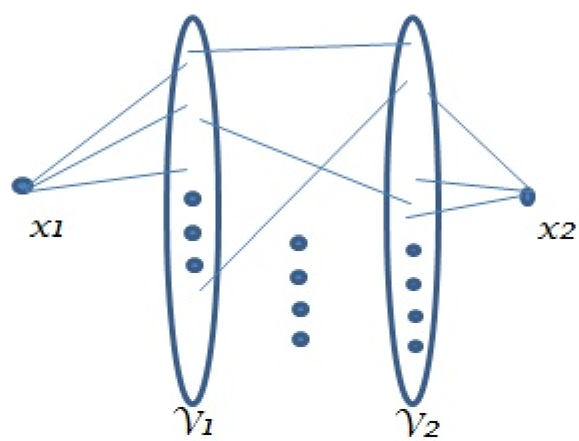

Figure 1.4: Menger's theorem being used. See Claim 1.4

with length less than equal to k. Otherwise we will get a section of the cycle C for which we will get the same contradiction.

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1.3 Homework

It is expected from the people to prove all variations of Menger's theorem(for edges as well as vertices. And for each both directed and undirected cases).