CSL851: Algorithmic Graph Theory **Semester I 2013-14**

Lecture 14: September 16

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

This lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

14.1 Some theorems and stuff

Proposition 14.1 *Let $\alpha(n)$ be a function such that $\alpha(n) \to \infty$ as $n \to \infty$, then for a random graph $G(n,p)$ it holds*

$$pN - \alpha(n)n \leq e(G) \leq pN + \alpha(n)n$$

almost asymptotically surely (a.a.s).

Proof: Let

$$X = \sum_{e \in 2^v} X_e$$
$$X = Np$$
$$Var(X) = Npq$$

where v is the number of vertices, and p and q holds their standard meanings ($q = 1 - p$)
Applying Chebyshev's inequality,

$$P(|X - Np|) \geq \alpha(n)n) \leq \frac{Var(X)}{(\alpha(n))^2 n^2}$$
$$P(|X - Np|) \geq \alpha(n)n) \leq \frac{Npq}{(\alpha(n))^2 n^2}$$
$$P(|X - Np|) \geq \alpha(n)n) \leq \frac{pq}{(\alpha(n))^2}$$

where pq is a constant. Thus, $\frac{pq}{\alpha(n)^2} \to 0$ as $n \to \infty$.

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14.2 Thresholds for connectivity

Definition 14.2 *Let Q be a graph property and let r, s, $t : N \to \mathbb{R}$ be functions. We say that*

1. *t is a lower threshold function for the graph property Q in the random graph model G(n,p) if for any* $p \leq t$ $\forall n$ *G(n,p) doesn't have property Q a.a.s*
2. *s is an upper threshold function for the graph property Q in the random graph model G(n,p) if for any* $p \geq s$ $\forall n$ *G(n,p) doesn't have property Q a.a.s*

Note that s and t may coincide. Without loss of generality we can assume that $t \leq s$*. Our aim is to bridge the gap between t and s to get better results.*

We need a technical result which handles the situation of non-independent random variables. Let $X_1, \ldots, X_k$ be random variables. Define

$$\Delta_1 = \sum_{\substack{i,j \in \{i,\ldots,k\} \\ i \neq k}} E(X_i, X_j)$$

We say $X_i \sim X_j$, $i \neq j$ if X_i and X_j are not independent.

$$\Delta_2 = \sum_{\substack{i,j \in \{i,\ldots,k\} \\ i \sim k}} E(X_i, X_j)$$

Theorem 14.3 *Let* $X_1, \ldots, X_k$ *be random variables (not necessarily independent) and let* $X = \sum_{i=1} kX_i$*. We have*

1. $Var(X) = \sum_i kVar(X_i) + \sum_{\substack{i,j \in \{i,\ldots,k\} \\ i \neq k}} Cov(X_i, X_j)$
2. *If* X_i*'s are Bernoulli random variables then*
   (a) $Var(X) = E(X) - E(X)^2 + \Delta_1$
   (b) $Var(X) \leq E(X) + \Delta_2$
   (c) $P(X = 0) \leq \frac{1}{E(X)} - 1 + \frac{\Delta_1}{(E(X))^2}$
   (d) $P(X = 0) \leq \frac{1}{E(X)} + \frac{\Delta_2}{(E(X))^2}$

Note: (a) and (b) can be concluded using straight-forward calculations. (c) and (d) can be derived from (a) and (b) respectively, using Chebyshev's inequality.

Theorem 14.4 *Let*

$$s(n) = \frac{ln(n) - \alpha(n)}{n}$$
$$t(n) = \frac{ln(n) + \alpha(n)}{n}$$

where $\alpha(n) \to \infty$ *as* $n \to \infty$ *and* $\alpha(n) \leq ln(n)$*.*
s is a lower threshold function and t in an upper threshold function for the property that G(n,p) has no isolated vertices.

Proof:

1. Upper threshold
   Consider $p \geq t$
   We proved that the probability that G(n,p) has isolated vertices is at most $n(1-p)^(n-1)$. This approaches zero as $n \to \infty$ when p is constant. But here p is a function of n. Now,

$$n(1-p)^{(n-1)} \leq n(1-t)^{(n-1)} \leq \frac{n}{1-t}(1-t)^n \leq \frac{n}{1-t}e^{(-t)}$$
$$n(1-p)^{(n-1)} \leq \frac{n}{1-t}e^{(-ln(n)-\alpha(n)} \leq \frac{1}{1-t}e^{-\alpha(n)}$$

   As $n \to \infty$, $t \to 0$.

$$\frac{1}{1-t}e^{-\alpha(n)} \to 0$$

   Hence proved.

2. Lower threshold
   Let $p \leq s$. Let X be the random variable which counts the number of isolated vertices. We need to prove that

$$P(X=0) \leq \frac{1}{E(X)} - 1 + \frac{\Delta_1}{(E(X))^2}$$

   and further prove that R.H.S $\to 0$ as $n \to \infty$.

$$E(X) = n(1-p)^{(n-1)} \geq n(1-s)^{(n-1)} = \frac{n}{1-s}(1-s)^n \geq \frac{n}{1-s}e^{-n(s+s^2)}$$

   (Using $1-x \geq e^{(-x-x^2)} for x \in [0,1/3]$)

$$E(X) = \frac{1}{1-s}e^{\alpha(n)+\beta(n)}, \text{ where } \beta(n) = \frac{ln^2(n)-2ln(n)+\alpha(n)}{n}$$

   Now, $\beta n \to 0$ as $n \to \infty$
   $\implies E(X) \to \infty$ as $n \to \infty$
   $\implies \frac{1}{E(X)} \to 0$ as $n \to \infty$
   Now consider, $\frac{\Delta_1}{E(X)^2}$

$$\Delta_1 = \sum_{v,w \in V v \neq w} P(X_v = 1 | not X_w = 1)$$
$$\implies \Delta_1 = \sum_{v,w \in V v \neq w} (1-p)^{2(n-2)+1}$$

   Using the fact that $(1-p)^{2(n-2)+1}$ edges are likely to exist if no edge exists with a vertex either from v or w.

$$\Delta_1 = n(n-1)(1-p)^{2n-3}$$
$$\frac{\Delta_1}{E(X)^2} = \frac{n(n-1)(1-p)^{2n-3}}{n^2(1-p)^{2(n-1)}} = (1-\frac{1}{n}) * \frac{1}{1-p}$$

   This tends to 1, as $n \to \infty$.
   Therefore, we have $\frac{1}{E(X)} - 1 + \frac{\Delta_1}{(E(X))^2} \to 0$ as $n \to \infty$
   Hence Proved.

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