CSL851: Algorithmic Graph Theory **Spring 2013**

Lecture 6: September 24

Lecturer: Anand Srivastav *Scribes: Devashish Tyagi*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

6.1 Threshold for Subgraph $G(n,p)$

Chebyshev's Inequality gives

$$P(X=0) \leq \frac{Var(X)}{E(X)^2} \leq \frac{E(X)+\Delta}{E(X)^2} = \frac{1}{E(X)} + \frac{\Delta}{E(X)^2}$$

provided that $X = \sum_{i=1}^{m} X_i$ and the $X_i's$ are 0/1 Random Variables and $\Delta = \sum_{0 \leq i < j \leq m} X_i X_j$.

Proposition 6.1

$$\Delta = \sum_{i,j \in [m], i \neq j, X_i \sim X_j} P(A_i \wedge A_j)$$

with A_i reprenting event $X_i = 1$. This equation can be rewritten as

$$\Delta = \sum_{X_i \sim X_j} P(A_i \wedge A_j) = \sum_i P(A_i) \sum_j P(A_j \mid Ai) = \sum_i P(A_i)\Delta_i^*$$

If $\Delta_i^* = \Delta^*$ then

$$\Delta = E(X)\Delta^*$$

Corollary 6.2 *If $E(X) \to \infty$ and $\frac{\Delta^*}{E(X)} \to 0$, then $X > 0$ a.a.s.*

Proof:

$$P(X=0) \leq \frac{1}{E(X)} + \frac{\Delta}{E(X)^2} = \frac{1}{E(X)} + \frac{\Delta^*}{E(X)}$$

In the above identity $\frac{1}{E(X)} \to 0$ and also $\frac{\Delta^*}{E(X)} \to 0$ when $E(X) \to \infty$. Therefore, $P(X=0) \to 0$. Hence proved $X > 0$ when $E(X) \to \infty$. ■

Definition 6.3 *Let H be a graph with v vertices and e edges . Let's call $\rho(H) = \frac{e}{v}$ density of H. We call H balanced if every subgraph $H^{'}$ (not necessarily induced subgraph) has $\rho(H^{'}) \leq \rho(H)$. H is strictly balanced, if any proper subgraph $H^{'}$ has $\rho(H^{'}) < \rho(H)$.*

Theorem 6.4 *Let H be a balanced graph with v vertices and e edges . Let A be the event that H is a subgraph of graphs from $G(n,p)$. Then $p = n^{-v/e}$ is the threshold funtion for A.*

Proposition 6.5 *We define LTF (Lower threshold function) and UTF (Upper threshold function). We say $r = T(n)$ is a threshold function for some property of $G(n,p)$, if the property does not hold a.a.s. if $p << T$ (i.e. $p/T \to 0$ as $n \to \infty$) and the property does hold if $p >> T$ (i.e. $p/T \to \infty$ as $n \to \infty$).*

Proof: Let S be a set of $|S| = v$ vertices in $G(n,p)$ ($v << n$). Let A_S be the event that a graph G from $G(n,p)$ restricted to S, which is G/S contains H as a subgraph. Then

$$p^e \leq P(A_S) \leq v!p^e \tag{6.1}$$

Let X_S be the indicator variable for A_S. 1 when A_S holds and 0 otherwise. Define $X = \sum_{|S|=v} X_S$. We now have to show $P(X = 0) \to 0$ as $n \to \infty$ so $X > 0$ a.a.s. To show this we invoke Corollary 6.2 $P(X \leq 0) \leq \frac{1}{E(X)} + \frac{\Delta^*}{E(X)}$ compute $E(X)$ and Δ^* and show $E(X) \to \infty$ and $\frac{\Delta^*}{E(X)} \to 0$ as $n \to \infty$.

$$E(X) = \sum_{|S|=v} E(X_S) = P(A_S)\binom{n}{v} = \theta(P(A_S)n^v)$$

We now use 6.1
if $p << n^{-v/e}$ then $E(X) \to 0$
if $p >> n^(-v/e)$ then $E(X) \to \infty$

We now have to compute Δ^*. $X_S \sim X_T$ if they are independent by definition. We write for abbreviation $S \sim T$. This happens if and only if $S \neq T$ and S and T have some common edges or if and only if $|S \cap T| = i$ with $2 \leq i \leq v-1$. Now fix S

$$\Delta^* = \sum_{T \sim S} P(A_T \mid A_S)$$

For each i there are $O(n^{v-1})$ choices of T. Fix S, T and consider $P(A_T \mid A_S)$. There are $O(1)$ ($v!$ is constant) possible copies of H and T. Since H is balanced each has at most ei/v edges in S. Then atleast there are $e - ei/v$ other edges. So,

$$\Delta^* = \sum_{i=2}^{v-1} O(n^{v-1}p^{e-ei/v}) = \sum_{i=2}^{v-1} O((n^{v-1}p^e)^{1-i/v}) = \sum_{i=2}^{v-1} O((n^{v-1}p^e)) = O(E(X))$$

so $\frac{\Delta^*}{E(X)} \to 0$ as $n \to \infty$. ■