CSL851: Algorithmic Graph Theory **Semester I 2013-14**

Lecture 13: September 11

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This lecture's notes covers the last lecture on preliminary knowledge of probability needed to study random graphs and introduction to Random Graph Model & its properties.

13.1 Preliminaries continued

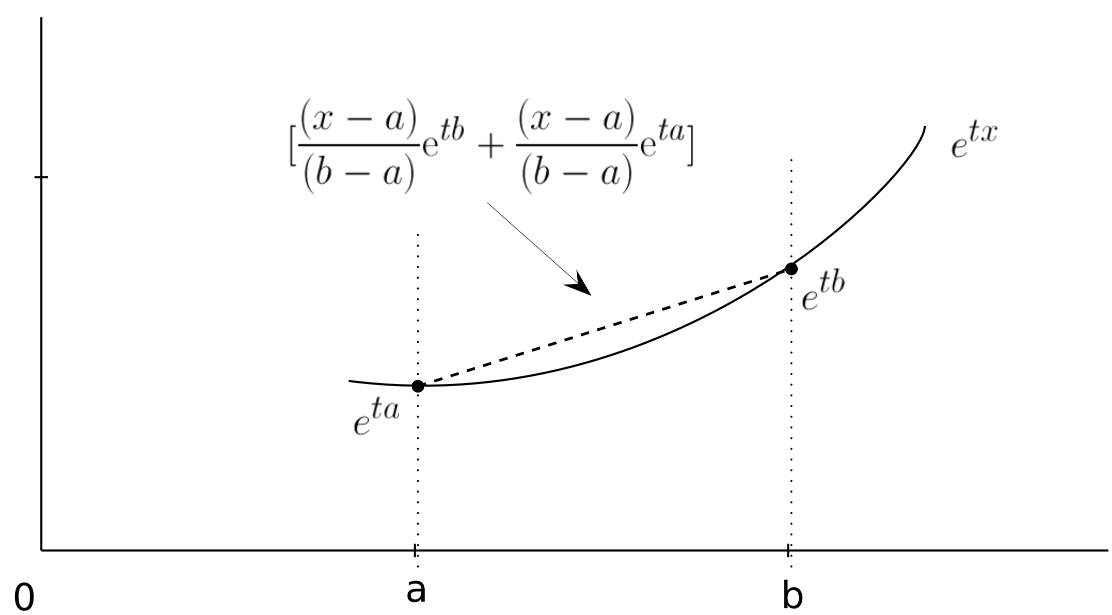
Lemma 13.1 *Let* $\mathbf{X}$ *be a random variable with* $\mathbf{a} \leq \mathbf{X} \leq \mathbf{b}$ *for some* $\mathbf{a}, \mathbf{b} \in \mathcal{R}$*. Suppose* $\mathbb{E}[\mathbf{X}] = 0$*, then for any* $t > 0$ *we have*

$$\mathbb{E}[e^{tX}] \leq e^{\frac{t^2(b-a)^2}{8}} \tag{13.1}$$

Here $\mathbb{E}[e^{tX}]$ is also known as the Moment Generating Function.

Proof: The function e^{tx} is convex on x. Thus, as it can be seen from the graph below, for $x \in [a, b]$ we have,

$$e^{tx} \leq [\frac{(x-a)}{(b-a)}e^{tb} + \frac{(x-a)}{(b-a)}e^{ta}] \tag{13.2}$$



Now, taking expectation of both sides we get,

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{tx}] &\leq \mathbb{E}[\frac{(x-a)}{(b-a)}\mathrm{e}^{tb} + \frac{(x-a)}{(b-a)}\mathrm{e}^{ta}] \\
&\leq \frac{(\mathbb{E}[x]-a)}{(b-a)}\mathrm{e}^{tb} + \frac{(\mathbb{E}[x]-b)}{(b-a)}\mathrm{e}^{ta} \qquad (13.3) \\
&\leq \frac{b\mathrm{e}^{ta} - a\mathrm{e}^{tb}}{b-a} \qquad (13.4)
\end{aligned}$$

Now consider, suitable function $h(t)$ s.t.

$$\mathrm{e}^{h(t)} = \frac{b\mathrm{e}^{ta} - a\mathrm{e}^{tb}}{b-a} \qquad (13.5)$$

It can be seen that for

$$f(x) = -px + \ln((1-p) + p\mathrm{e}^{x})$$

where,

$$p = \frac{a}{b-a} \qquad (13.6)$$

we get for $\hat{t} = t(b-a)$ and $h(t) = f(\hat{t})$, h(t) satisfies the the eq. 13.4
we are done if $h(t) \leq \dfrac{t^2(b-a)^2}{8}$, which can be proven using taylor expansion of f at 0.

$$h(t) = f(\hat{t}) = f(0) + \frac{f`(0)}{1!}\hat{t} + \frac{f``(\eta)}{2!}\hat{t}^2 \textbf{ : } \eta \in [0, \hat{t}]$$

Now, $\mathbf{f(0) = 0}$, $\mathbf{f`(0) = 0}$) and $\mathbf{f''(x)} = \dfrac{\mathbf{p(1-p)e^x}}{\mathbf{(p+(1-p)e^x)^2}}$
As $h(t) = \dfrac{f``(\eta)}{2}\hat{t}^2$, It will be enough to show that $\mathbf{f``(x)} \leq \dfrac{\mathbf{1}}{\mathbf{4}}$
as then, $h(t) \leq \dfrac{\hat{t}^2}{8} = \dfrac{t^2(b-a)^2}{8}$

$f''(x) = \dfrac{p(1-p)e^x}{(p+(1-p)e^x)^2}$ can be written as $\dfrac{\alpha\beta}{(\alpha+\beta)^{\mathbf{2}}}$ *for* $\alpha = p, \beta = (1-p)e^x$
By using A.M. $\geq$ G.M.we get $\sqrt{\alpha\beta} \leq \dfrac{\alpha+\beta}{2} \Rightarrow \alpha\beta \leq \dfrac{(\alpha+\beta)^2}{4}$ Thus proved ■

Proof of theorem 12.9 **Hoeffdings inequality** Let independent and indentically distributed random variables $X_i : 1 \leq i \leq n$, be uniformly bounded as $a_i \leq X_i \leq b_i$, $1 \leq i \leq n$. Also, let $X = \sum_{i=1}^{n} X_i$ and $c_i = b_i - c_i$. Then for $\lambda \geq 0$

$$[P((X - \mathbb{E}[X]) \geq \lambda) \leq 2e^{\dfrac{-2\lambda^2}{\sum_{i=1}^{n} c_i}} \tag{13.7}$$

Proof:: Let $t > 0$, then using $g(x) = e^{tx}$ in Markov's inequality (Theorem 12.7) we get

$$\begin{aligned}
\mathbb{P}(X - \mathbb{E}[X] \geq \lambda) &\geq e^{-t\lambda}\mathbb{E}[e^{t(X-\mathbb{E}[X])}] \\
&= e^{-t\lambda}\mathbb{E}[e^{t\sum_{i=1}^{n} c_i}] \\
&= e^{-t\lambda}\mathbb{E}[e^{t\sum_{i=1}^{n} X_i - \mathbf{E}[X_i]}] \\
&= e^{-t\lambda}\mathbb{E}[\prod_{i=1}^{n} e^{tX_i - \mathbf{E}[X_i]}] \\
&= e^{-t\lambda}\prod_{i=1}^{n} \mathbb{E}[e^{tX_i - \mathbf{E}[X_i]}] \\
&\leq e^{-t\lambda}.e^{\frac{1}{8}t\sum_{i=1}^{n}(b_i-a_i)^2]} \quad \textit{using lemma 12.1 proved above} \\
&\leq e^{-t\lambda+\frac{t^2}{8}+\sum_{i=1}^{n}(c_i)^2]} \quad \textit{solving quadratic in } t \textit{ for minima} \\
&\leq e^{\frac{-2\lambda^2}{\sum_{i=1}^{n}(c_i)^2}}
\end{aligned} \tag{13.8}$$

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13.2 Basics of Random Graph Model G(n,p)

Let $p \in [0,1]$ and let $V = \{v_1, v_2 ... v_n\} \simeq \{1, 2.., n\}$ be set of nodes in a graph. For every potential edge $e = (i, j)$ toss a p-biasied coin independently for all edges to decide whether edge e(i,j) is in graph or not. The outcome of this random experiment is a graph and this generation process **G(n,p)** is called Random Graph Model. The random graph is genrated by sequence of p-bernouli random variable $X(_e$ for $e \in 2^V$. Formally, $G(n,p)$ is finite probability space, with sample space Ω being set of all graphs on V, for a graph **G**

$$\mathbf{P}(\{G\}) = p^{|E(G)|}(1-p)^{N-|E(G)|} \tag{13.9}$$

Here E(G) is set of edges of G and $N = \dfrac{n(n-1)}{2} = |2^V|$

Definition 13.2 *Let **Q** be a graph property. We say that graph from G(n,p) has a property Q assymototically almost surely (a.a.s)*
if

$$\lim_{n\to\infty} \mathbf{P}(\textit{graph from G(n,p) has property Q}) \longrightarrow 1$$

Proposition 13.3 *Graph from G(n,p) do not have isolated vertices is assymototically almost surely (a.a.s)*

Proof: For $v \in V$, define R.V

$$X_v = \begin{cases} 1 & : \text{if } \mathbf{v} \text{ is isolated} \\ 0 & : \text{otherwise} \end{cases} \tag{13.10}$$

Let,

$$X = \sum_{v \in V} X_v$$

Then desired event is $X = 0$, while bad event being $X \geq 1$ Now,

$$\mathbf{E}[X_v] = \mathbb{P}(X_v = 1) = (1-p)^{n-1}$$

And,

$$\begin{aligned} \mathbf{P}(X \geq 1) &\leq \frac{\mathbf{E}[X]}{1} \\ &= \mathbf{E}[\sum_{v \in V} X_v] \\ &= \sum_{v \in V} \mathbf{E}[X_v] \\ &= n(1-p)^{n-1} \end{aligned} \tag{13.11}$$

Now using L-Hospital rule it can be shown that

$$\lim_{n \to \infty} n(1-p)^{n-1} \longrightarrow 0$$

by writting it as

$$\begin{gathered} \lim_{n \to \infty} \frac{n}{(1-p)^{1-n}} \\ \lim_{n \to \infty} (1-p)^{n-1} . \ln(1-p) \end{gathered}$$

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13.2.1 Further reading

L'Hospital's Rule `http://mathworld.wolfram.com/LHospitalsRule.html`