CSL 851: Algorithmic Graph Theory **Fall 2013**

Lecture 2: Chordal Graphs

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

In this lecture we would study about Chordal Graphs.

2.1 Induced Subgraph

Definition 2.1 *Let $G = (V, E)$ be a Graph.*

Let $V' \subseteq V$ be a subset of vertices of G.

The subgraph of G induced by V' is the subgraph $G' = (V', E')$ of G that has $E' = E \cap (V' \times V')$.

That is, it contains all the edges of G that connect elements of the given subset of the vertex set of G and only those edges.

2.2 Chordal Graphs

Definition 2.2 *A Chordal Graph is a graph that does not contain an induced cycle of length greater than* 4.

In other words, it is a graph in which every cycle of length four and greater has a cycle chord.

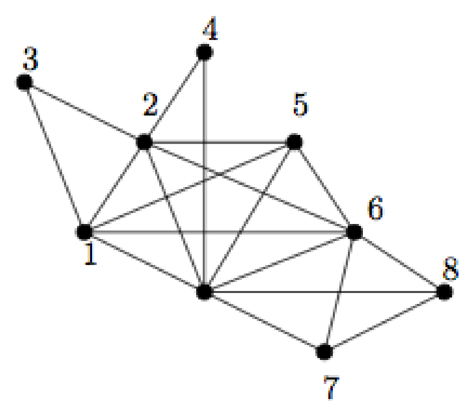

Figure 2.1: A chordal graph

Theorem 2.3 *A graph G is chordal iff it has a perfect elimination ordering.*

Proof: The easy part is to show that if G has a perfect elimination ordering, then it is chordal. Suppose, for contradiction, that this is false. Let G be a graph with a perfect elimination ordering and suppose there is a chordless cycle v_1,v_2,. . . ,v_l of length $l \geq 4$ in G. Let v_i be the vertex in the cycle that occurs first in the perfect elimination ordering. Then v_{i-1} and v_{i+1} are neighbors of v_i in G that occur later in the ordering. Since the ordering is perfect, there must be an edge between v_{i-1} and v_{i+1}, but this contradicts the assumption that the cycle is chordless.

Now, show the converse, that if G is chordal then it has a perfect elimination ordering. For that we would need the concept of seperators.

Definition 2.4 *A separator is a partition $V = S \cup A \cup B$ of the vertices such that there are no edges between A and B.*

Definition 2.5 *Given two non-adjacent vertices a and b, an (a, b)-separator is a separator $V = S \cup A \cup B$ such that $a \in A$ and $b \in B$.*

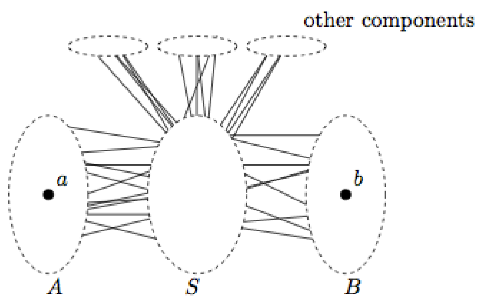

Figure 2.2: A (a, b)-separator

Definition 2.6 *Given two non-adjacent vertices a and b, a minimal (a, b)-separator is an (a, b)-separator $V = S \cup A \cup B$ such that no subset of S is an (a, b)-separator.*

Definition 2.7 *A simplicial vertex of a graph G is a vertex v such that the neighbours of v form a clique in G.*

Lemma 2.8 *Given a chordal graph $G = (V, E)$ and two vertices $a, b \in V$ such that $(a, b) \notin E$, any minimal a-b separator is a clique.*

Proof: We would prove this by contradiction. Let S be a minimal (a, b)-separator. For any vertex set T , let G_T be the graph induced by T . Then G_{V-S} has a number of connected components; one contains a (let those vertices be A), one contains b (let those vertices be B), and there may be other connected components.

Consider any two vertices x, y in the minimal a-b seperator S and suppose that $(x, y) \notin E$.

Note first that x must have a neighbour a_x in A, for otherwise $S - x$ would also be an (a,b)-separator, contradicting the minimality of S. Likewise, y has a neighbour a_y in A.

Since G_A is connected, there is a path from a_x to a_y using only vertices in A. Thus, there exists a path from x to y for which all intermediate vertices are in A. Among all such paths, let the shortest one be x,a_1,a_2,. . . ,a_k,y and note that it has length at least 2 since x and y are not adjacent. Similarly we can find a shortest path from x to y for which all intermediate vertices are in B.

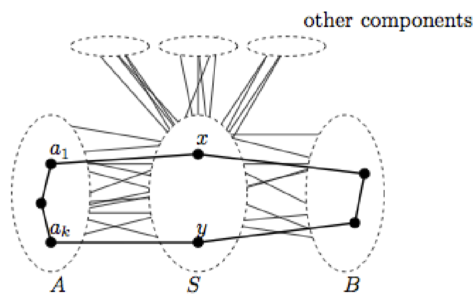

Figure 2.3: Minimal (a,b)-separator is a clique

Combining the two paths yields a cycle of length at least 4, which must have a chord since G is chordal. However, there is no chord in the cycle from x or y to either A or B since we chose the shortest paths from x to y in each component. Neither is there an edge from A to B since A and B are two different components. The only other possibility is for there to be a chord between x and y, but x and y are not adjacent. So we have a contradiction, which means that $(x, y) \in E$. ■

Clearly, if G has a perfect elimination order, then the last vertex in it is simplicial in G. This gives rise to a simple algorithm to find a perfect elimination order if one exists:

Algorithm: Find perfect elimination order.
For i = n, . . . , 1
Let G_i be the graph induced by V v_{i+1},. . . ,v_n.
Test whether Gi has a simplicial vertex v.
If no, then stop. G_i (and therefore G) has no perfect elimination order.
Else, set $v_i = v$.
v_1,. . . ,v_n is a perfect elimination order.

Note that if G is chordal, then after deleting some vertices, the remaining graph is still chordal. So in order to show that every chordal graph has a perfect elimination order, it suffices to show that every chordal has a simplicial vertex; the above algorithm will then yield a perfect elimination order.

Now we show that every chordal graph has a simplicial vertex. In fact, we show a slightly stronger statement, which is needed for the induction hypothesis.

Lemma 2.9 *A connected chordal graph is either a clique, or it contains two non adjacent simplicial vertices.*

Proof: If G is chordal and it is a clique we are done. Assume that it is not a clique. Therefore we have two non-adjacent vertices a, b in G. Consider the minimal $a - b$ separator,S.
Induction on $A \cup S$(refer to the definition above).
If it is a clique then a is a simplicial vertex or it has two non adjacent simplicial vertices a_1 and a_2, both of which cannot lie in S as S is a clique. Therefore either a_1 or a_2 lie in A. Similarly we can find a second non adjacent simplician vertex when we consider B. ■

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2.3 Independent Set

The maximal independent set problem is a NP-Hard Problem on general graphs but on graphs having a partial elimination ordering this problem can be solved efficiently.

Claim 2.10 *There is an efficient algorithm to solve the independent set problem on graphs with a partial elemination ordering.*

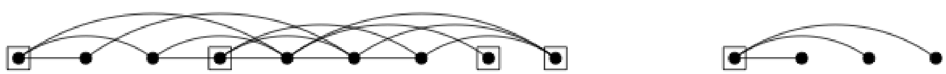

Figure 2.4: Two graphs with a vertex order and the result of the greedy algorithm for Independent Set.

Proof: Algorithm: Scan the vertices in order, and for each v_i, add v_i to I if none of its predecessors has been added to I.
Scan Order: Let v_1, v_2,..., v_n be a perfect elimination order. Then the greedy algorithm applied with order v_n,v_{n-1},...,v_1 gives a maximum independent set. ■