CSL 851: Algorithmic Graph Theory **Fall 2013**

Lecture 1: October 9

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

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We study special classes of Graphs that are more complex than a tree but simpler than a general graph. The main motivation is to solve NP-Hard optimization problems such as Vertex Cover, Clique, Independent Set etc. on these graphs in polynomial time. We will define Interval Graphs, Perfect Elimination Ordering and finally solve the Coloring problem on Interval Graphs.

1.1 Interval Graphs

Definition 1.1 *A graph is said to be an interval graph if*

- *With every vertex we can associate an interval(open/closed) on real line.*
- *Two vertices share an edge iff the corresponding intervals are not disjoint.*

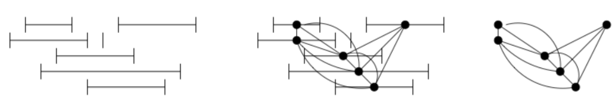

Figure 1.1: Interval Graphs

Lemma 1.2 *Given an interval graph with n vertices, one can assign intervals to vertices such that every end point is a distinct integer from 1 to 2n.*

Proof:

Consider a point x where two or more intervals begin/end. We classify these intervals into 4 classes:

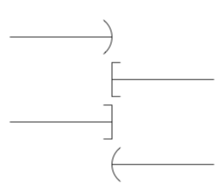

Figure 1.2: Interval Classification

1. Intervals ending at x and open at x.
2. Intervals starting at x and closed at x.
3. Intervals ending at x and closed at x.
4. Intervals starting at x and open at x.

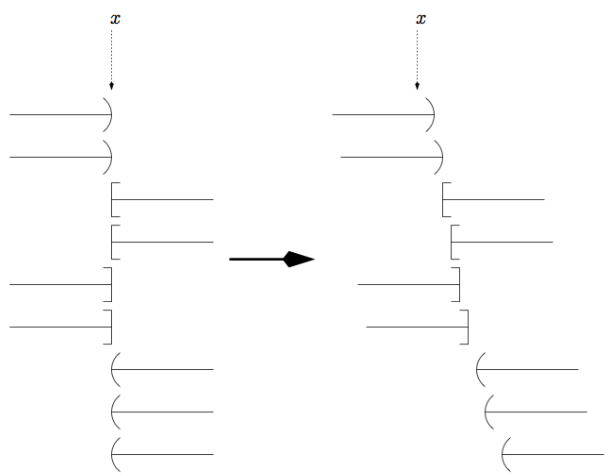

Figure 1.3: Interval Shifting

Let ϵ be the minimum distance amongst all pairs of distinct end points. As seen in the figure, shifting intervals in increasing order of their class keeping all shifts less than ϵ serves our purpose. It is trivial to see that all adjacency relations are still maintained. Now that all intervals have distinct end points, these end points can be assigned integers from 1 to 2n.

■

1.2 Perfect Elimination Ordering

Definition 1.3 *A graph has a P.E.O. if one can order the vertices such that for every vertex, all its neighbours that lie to the left of it (in the ordering) form a clique.*

Examples:

- For a Complete graph any ordering is a P.E.O. since for any vertex, all the vertices appearing before it are the left neighbours and they form a clique.
- For a Tree, vertices can be ordered be removing leaves and putting in a stack until the tree is empty. This way, every vertex will have only one left neighbour(any leaf has only one neighbour), which is a clique.

Lemma 1.4 *Any graph which contains a n-cycle, $n \geq 4$ as an induced subgraph does not have a P.E.O.*

Proof: Assume there is a P.E.O. Consider the vertex v of the cycle that appears rightmost in the ordering. Clearly its two neighbours in the n-cycle do not have an edge between them. But both these neighbours also lie to the left of v in the ordering. A contradiction. ■

Theorem 1.5 *Every interval graph has a P.E.O.*

Proof: Order vertices by left end point (all end points are distinct by previous result). In this ordering, given a vertex v, intervals of all its left neighbours must contain the left end point of v's interval (since they intersect the interval of v). This implies any two left neighbours of v must share an edge. Thus, they form a clique. ■

1.3 Graph Coloring

Definition 1.6 *With minimum number of colors, assign each vertex of a graph G a color so that each edge has end vertices of different colors. This number is called the chromatic number of the graph, $\chi(G)$.*

Lemma 1.7 $\chi(G) \geq \omega(G)$ *where* $\omega(G)$ *is the maximum clique size*

Proof: Each vertex in the largest clique requires a distinct color. ■

Theorem 1.8 *There is an efficient algorithm to solve graph coloring problem on graphs with P.E.O.*

Proof: Let us enumerate the set of colors with the set of natural numbers. Going in the P.E.O. order, for each vertex, pick the smallest numbered color not used by any of its left neighbour. Clearly, each vertex would have less than $\omega(G)$ neighbours(otherwise max. clique size would be greater.) Hence, atleast one of $1...\omega(G)$ colors must be free for every vertex. This implies $\chi(G) \leq$ number of colors used $\leq \omega(G)$. Using previous lemma, we thus prove that $\chi(G) =$ number of colors used $= \omega(G)$. ■