Tutorial Sheet 4

Aug 17, 19, 20

1. Determine whether each of these sets is countable or uncountable. For those that are countably, exhibit a one-to-one mapping from the set to the set of positive integers.
   (a) all positive rational numbers that cannot be written with denominators less than 4
   (b) the real numbers not containing 0 in their decimal representation
   (c) the real numbers containing only a finite number of 1s in their decimal representation
   (d) integers divisible by 5 but not by 7
   (e) the real numbers with decimal representations consisting of all 1s
   (f) the real numbers with decimal representations of all 1s or 9s.
2. Show that if A is an infinite set, then it contains a countably infinite subset.
3. Showthat there is no infinite set A such that $|A| < |Z^+| = \aleph_0$.
4. Show that the union of a countable number of countable sets is countable.
5. The Schroeder-Bernstein theorem states that if there is an injective mapping from A to B and an injective mapping from B to A then there exists a bijective mapping from A to B. Use this to argue that
   (a) $(0, 1)$ and $[0, 1]$ have the same cardinality
   (b) $(0, 1)$ and $\Re$ have the same cardinaility
6. Show that there is no one-to-one correspondence from the set of positive integers to the power set of the set of positive integers. [Hint: Assume that there is such a one-to- one correspondence. Represent a subset of the set of positive integers as an infinite bit string with i^{th} bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with itsi^{th} bit equal to the complement of the i^{th} bit of the i^{th} string in the list. Show that this new bit string cannot appear in the list.]
7. Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1. Use this to conclude that $\aleph_0 < |P(Z^+)| = |\Re|$.
8. Show that if S is a set, then there does not exist an onto function f from S to$P(S)$, the power set of S. Conclude that $|S| < |P(S)|$. This result is known as Cantor's theorem. [Hint: Suppose such a function f existed. Let$T = s \in S | s \notin f(s)$ and show that no element s can exist for which $f(s) = T$.]