Tutorial Sheet 2

Aug 3, 5, 6

1. Establish these logical equivalences, where *x* does not occur as a free variable in *A*. Assume that the domain is nonempty.
   a. $\forall x\,(A \to P(x)) \equiv A \to \forall x\, P(x)$
   b. $\exists x(A \to P(x)) \equiv A \to \exists x P(x)$
2. Find a common domain for the variables *x*, *y*, and *z* for which the statement $\forall x \forall y\ \big((x \neq y) \to \forall z((z = x) \lor (z = y))\big)$ is true and another domain for which it is false.
3. Assuming all quantifiers have the same nonempty domain show that
   a. $\forall x\, P(x) \land \exists x Q(x)$ is logically equivalent to $\forall x \exists y(P(x) \land\ Q(y))$
   b. $\forall x P(x) \lor \exists x Q(x)$ is equivalent to $\forall x \exists y\ (P\ (x) \lor Q(y))$
4. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.
5. Let $P(x), Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x), Q(x)$, and $R(x)$.
   a. All clear explanations are satisfactory.
   b. Some excuses are unsatisfactory.
   c. Some excuses are not clear explanations.
   d. Does (c) follow from (a) and (b)?
6. Let $F(x, y)$ be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.
   a. Everybody can fool somebody.
   b. There is no one who can fool everybody.
   c. Everyone can be fooled by somebody.
   d. No one can fool both Fred and Jerry.
   e. Nancy can fool exactly two people.
7. Use rules of inference to show that if $\forall x(P(x) \lor Q(x)), \forall x(\neg Q(x) \lor S(x)), \forall x(R(x) \to \neg S(x))$ and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.
8. Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j\ -\ k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.
9. Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [Hint: Work backward, assuming that you did end up with nine zeros.]
10. Prove or disprove that there is a rational number x and an irrational number y such that xy is irrational.
11. Prove that between every two rational numbers there is an irrational number.