Answers to Tutorial Sheet 2.

1. We consider two cases

   (a) A is False. Then both the equivalences reduce to True≡True which is a tautology.

   (b) A is True. Then $(A \rightarrow B) \equiv B$ and hence the first statement reduces to $\forall x P(x) \equiv \forall x P(x)$ which is a tautology. Similarly the second statement reduces to $\exists x P(x) \equiv \exists x P(x)$ which is again a tautology.

2. Let D be a domain for the quantifiers. The statement says that if we consider two distinct elements in the domain then all other elements in the domain should be one of thes two elements. This would therefore be true if $|D| = 2$. However if $|D| > 2$ then the statement is False.

3. To show $A \equiv B$ we have to show $(A \rightarrow B) \wedge (B \rightarrow A)$.Further, to show $A \rightarrow B$, we need only argue that if A is TRUE then B is also TRUE.

   (a) $\forall x P(x) \bigwedge \exists x Q(x)$ implies $\forall x P(x)$ and $Q(c)$ for some c. This implies $\forall x(P(x) \wedge Q(c))$ which in turn implies $\forall x \exists y(P(x) \wedge Q(y))$.To prove the converse note that $\forall x \exists y(P(x) \wedge Q(y))$ implies there is a c such that $Q(c)$ and for all x in the domain $P(x)$. Hence $\forall x P(x) \wedge Q(x)$which in turn implies $\forall x P(x) \bigwedge \exists x Q(x)$.

   (b) $\forall x P(x) \vee \exists x Q(x)$ implies $\forall x P(x)$ or $Q(c)$ for some c. This implies $\forall x(P(x) \vee Q(c))$ which in turn implies $\forall x \exists y(P(x) \vee Q(y))$.To prove the converse note that $\forall x \exists y(P(x) \vee Q(y))$ implies there is a c such that $Q(c)$ or for all x in the domain $P(x)$. Hence $\forall x P(x) \vee Q(x)$which in turn implies $\forall x P(x) \vee \exists x Q(x)$.

4. $\forall n \exists i \exists j \exists k \exists l((n > 0) \rightarrow (n^2 = i^2 + j^2 + k^2 + l^2))$, where the domain of the quantifiers is the set of integers.

5. Recall the discussion in class for restricting the domain of a quantifier. Let $D = \{x | Q(x)\}$. Then the statement "For all $x \in D$, $P(x)$" can be written as $\forall x(Q(x) \rightarrow P(x))$ and the statement "there exists $x \in D$, $P(x)$" can be written as $\exists x(Q(x) \wedge P(x))$. Using these

   (a) $\forall x(P(x) \rightarrow Q(x))$

   (b) $\exists x(R(x) \wedge \neg Q(x))$

   (c) $\exists x(R(x) \wedge \neg P(x))$

   (d) Yes. Let c be sentence for which $R(c) \wedge \neg Q(c)$ is True. This is equivalent to$Q(c) \rightarrow R(c)$. From (a) it follows that $P(c) \rightarrow Q(c)$. Hence $P(c) \rightarrow R(c)$ which is equivalent to $R(c) \wedge \neg P(c)$. By existential generlization (c) follows.

6. Solutions

   (a) $\forall x \exists y F(x, y)$,

   (b) $\neg(\exists x \forall y F(x, y)) \equiv \forall x \exists y \neg F(x, y)$,

   (c) $\forall x \exists y F(y, x)$,

   (d) $\neg(\exists x(F(x, "Fred") \wedge F(x, "Jerry")))$,

   (e) $\exists x \exists y(F("nancy", x) \wedge F("nancy", y) \wedge \forall z((z \neq x) \wedge (z \neq y) \rightarrow \neg F("nancy", z)))$.

7. Since $\exists x \neg P(x)$ is true, let c be such that $P(c)$ is False. This together with $P(c) \vee Q(c)$ implies $Q(c)$ is True. Since $\neg Q(c) \vee S(c)$, we conclude $S(c)$ is True. Since $R(c) \rightarrow \neg S(c)$, this implies $R(c)$ is False. Hence $\exists x \neg R(x)$ is True.