Solutions to Tutorial sheet 6

Note Title 14-04-2015

1) No. of ternary strings containing "00" or "11".

Let a_n = no. of such strings which begin with 0

b_n = ______ with 1

c_n = ______ with 2

Consider strings with which begin with 0. Three possibilities for 2nd character a) 0 — here are 3^{n-2} such strings

b) 1 — there are b_{n-1} such strings

c) 2 — there are c_{n-1} such strings

Hence $a_n = b_{n-1} + c_{n-1} + 3^{n-2}$

Similarly $b_n = a_{n-1} + c_{n-1} + 3^{n-2}$.

& $c_n = a_{n-1} + b_{n-1} + c_{n-1}$

Thus $a_n + b_n + c_n = 2(a_{n-1} + b_{n-1} + c_{n-1}) + c_{n-1} + 2.3^{n-2}$

Let $p_n = a_n + b_n + c_n$ be the # of such strings of length n

Then $c_{n-1} = p_{n-2}$

& hence $p_n = 2p_{n-1} + p_{n-2} + 2.3^{n-2}$

Sanity check: $p_0 = 0$, $p_1 = 0$, $p_2 = 2$, $p_3 = |\{001, 011, 110, 112, 100, 002, 200, 211, 000, 111\}| = 10$

$p_2 = 2.0 + 0. + 2.3^0 = 2$ ✓

$p_3 = 2.p_2 + p_1 + 2.3^1 = 4 + 0 + 6 = 10$ ✓

Q2) # of strings that contain "01". Let a_n be the no. of such bit strings which begin with 0 & b_n be the no. which begin with 1.

If string begins with 0, there are 2 possibilities for 2^{nd} character

- 0 — there are a_{n-1} such strings
- 1 - there are 2^{n-2} such strings

Thus $a_n = a_{n-1} + 2^{n-2}$

Similarly $b_n = b_{n-1} + a_{n-1}$ & hence

$$a_n + b_n = 2(a_{n-1} + b_{n-1}) + 2^{n-2} - b_{n-1}$$

If p_n is the no of such strings then $p_n = 2p_{n-1} + 2^{n-2} - b_{n-1}$

But $b_{n-1} = p_{n-2}$ & so

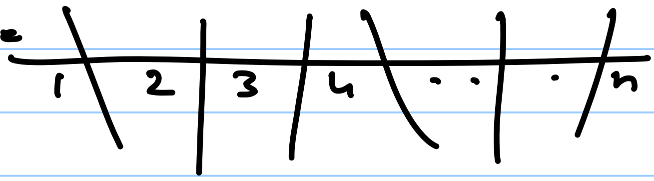
$$p_n = 2p_{n-1} - p_{n-2} + 2^{n-2}$$

Now

$p_0 = 0$, $p_1 = 0$, $p_2 = 1$, $p_3 = 2p_2 - p_1 + 2^1 = 4$
$= |\{010, 011, 001, 101\}|$

$p_4 = 2p_3 - p_2 + 4 = 8 - 1 + 4 = 11 = |\{0100, 0101, 0110, 0111,$
$0010, 0011, 1010, 1011, 0001, 1001, 1101\}|$

Q3) The n^{th} line intersects all previous $n-1$ lines & is divided into n segments. Each of these segments is dividing an existing region into two regions. Hence addition of the n^{th} line creates n new regions & so



$$R_n = R_{n-1} + n$$

Note $R_0 = 1$, $R_1 = 2$, $R_2 = 4$ ....

Q4) To move n disks from peg 1 to peg 3, we will move $(n-1)$ disks from 1 to 3, the largest disk from 1 to 2, the $(n-1)$ disks from 3 to 1, the largest disk from 2 to 3 & finally the $n-1$ disks from 1 to 3. Thus if p_n denotes the no. of steps required to move n disks from 1 to 3 or from 3 to 1 (should be the same by symmetry) then we have

$$p_n = 3p_{n-1} + 2$$

Now $p_1 = 2$, $p_2 = 8$ can be verified by hand.

Q5) $A_n = \begin{bmatrix} 2 & 1 & & & & 0 \\ 1 & 2 & 1 & \ddots & & \\ & 1 & 2 & \ddots & 1 & \\ & & \ddots & 2 & \ddots & \\ & 0 & & 1 & \ddots & 1 \\ & & & & 1 & 2 \end{bmatrix}$

Expanding along row 1 we get

$$d_n = 2d_{n-1} - \det \begin{array}{c} \xleftrightarrow{\;n-1\;} \\ \begin{bmatrix} 1 & 1 & & \\ 0 & 2 & 1 & \\ & 1 & 2 & \ddots \\ & & 1 & \ddots \end{bmatrix} \end{array} \updownarrow n-1$$

$$= 2d_{n-1} - d_{n-2}$$

Check

$d_1 = 2, \quad d_2 = \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$

$d_3 = 2d_2 - d_1 = 6 - 2 = 4$

$$\det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 2 \times \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= 2 \times 3 - 2 = 4$$