Solutions to Sheet 8

1

Let $A_i, 1 \leq i \leq 3$, be the set of sequences in which didgit i does not appear. Then we are interested in $|\neg(A_1 \cup A_2 \cup A_3)|$. Now $|A_i| = 9^n$, $|A_i \cap A_j| = 8^n$ and $|A_1 \cap A_2 \cap A_3| = 7^n$. Hence $|(A_1 \cup A_2 \cup A_3)| = 3 \cdot 9^n - 3 \cdot 8^n + 7^n$ and so $|\neg(A_1 \cup A_2 \cup A_3)| = 10^n - 3 \cdot 9^n + 3 \cdot 8^n - 7^n$.

2

Let A_i be the set of sequences in which both copies of the i^{th}character of the alphabet occur together. Then we are interested in $|\neg(\cup_{i=1}^{26} A_i)| = \frac{52!}{2^{26}} - |\cup_{i=1}^{26} A_i|$. Now $|A_i| = \frac{51!}{2^{25}}$, $|A_i \cap A_j| = \frac{50!}{2^{24}}$ and $|\cap_{r=1}^{k} A_{i_r}| = \frac{(52-r)!}{2^{26-r}}$. So the required quantity is $\sum_{r=0}^{26} (-1)^r \binom{26}{r} \frac{(52-r)!}{2^{26-r}}$.

3

Interpreting this as the number of ways of choosing 13 cards from 52 such that at least one suit is missing. Let A_1, A_2, A_3, A_4be the sets of hands which have spades, clubs, hearts and diamonds missing. Then we are interested in $|A_1 \cup A_2 \cup A_3 \cup A_4|$. Now $|A_i| = \binom{39}{13}$, $|A_i \cap A_j| = \binom{26}{13}$, $|A_i \cap A_j \cap A_k| = 1$ and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$. Hence $|A_1 \cup A_2 \cup A_3 \cup A_4| = 4 \cdot \binom{39}{13} - 6 \cdot \binom{26}{13} + 4$.

4

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$and let $d|n$. Now $\mu(d) \neq 0$ if d is a product of distinct prime factors. If d is a product of r distinct prime factors then $\mu(d) = (-1)^r$and hence $\sum_{d|n} \mu(d) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r$. Since $(1+x)^k = \sum_{r=0}^{k} \binom{k}{r} x^r$, hence $\sum_{r=0}^{k} \binom{k}{r} (-1)^r = 0$.

5

The number of different outcomes is 6^8. Let A_i be the outcomes in which i does not appear. Then we are interested in $|\neg(A_1 \cup A_2 \cup \cdots \cup A_6)|$. Now $|A_i| = 5^8$, $|A_i \cap A_j| = 4^8$, $|A_i \cap A_j \cap A_k| = 3^8$ and $|A_i \cap A_j \cap A_k \cap A_l| = 2^8$. Then $|A \cup A_2 \cup \cdots \cup A_6| = 6 \cdot 5^8 - \binom{6}{2} \cdot 4^8 + \binom{6}{3} \cdot 3^8 - \binom{6}{4} \cdot 2^8 + \binom{6}{5} \cdot 1$, and so the number of outcomes in which all numbers appear is $6^8 - 6 \cdot 5^8 + 15 \cdot 4^8 - 20 \cdot 3^8 + 15 \cdot 2^8 - 6$.

6

We will first count the number of bit strings containing 5 consecutive zeros. Treat the first occurence of 5 consecutive zeros as a cluster, C. Then the 10 length bit string can be viewed as a 6 length string in which one character is C while the remaining are 0 or 1 and the character immediately preceeding C is a 1.

1. There are 2^5 such strings which begin with a C.
2. If C is not the first character then we can combine it with the preceding 1 to form a character C'. So now we have to count the number of 5 length strings in which one character is C' and the remaining are 0 or 1. The number of such strings is $2^4 \times 5 = 80$.

Hence there are $80 + 32 = 112$ strings with 5 consecutive zeroes. By symmetry the number of strings with 5 consecutive 1's is also 112 and since there are exactly 2 strings which contain both 5 consecutive 0's and 5 consecutive 1's, the required number of strings is 112+112-2=222.

7

A truth table for a proposition on n variables has 2^n rows, one for each of the 2^n choices of the input variables. The value of the proposition for each input is 0 or 1 and hence a truth table is uniquely specified by a 2^n-bit vector. Since there are 2^{2^n} such bit vectors that is the also the number of different truth tables.

8

Once we decide on the two neighbors of each person, there are exactly 2 ways of seating them on a circular table. The number of ways of seating r people around a circular table is $(r-1)!$. When two seating plans in which each person has the same two neighbors are considered indistinguishable, the number of distinct ways of seating r people reduces to $(r-1)!/2$. Since we have to choose these r people from a set of n, the total number of different possibilities is $C(n, r) * (r-1)!/2$.

9

Since $C(n,r) = C(n,n-r)$, the expression on the left can be rewritten as $\sum_{k=1}^{n} \binom{n}{k} \binom{n}{n-k+1}$ which can be interpreted as the number of ways of picking $n+1$ balls from n red and r blue balls where at least one red and one blue ball has to be included. The restriction of including one red/blue ball is no restriction and hence this is just the number of ways of picking$n+1$ balls from $2n$ balls which is $C(2n, n+1)$. Hence we have to show that $C(2n, n+1) = C(2n+2, n+1)/2 - C(2n, n)$. This is true since$C(2n, n+1) + C(2n, n) = C(2n+1, n+1) = C(2n+2, n+1)/2$.

10

After distributing one object to each of the 4 boxes we are left with 2 objects. These 2 objects can both go into the same box or into 2 different boxes yielding 2 ways of doing the distribution.

11

For any way of defining (A, B) such that$A \subseteq B \subseteq S$, each element of S is either in $A, B - A$, or $S - B$. Hence with very ordered pair (A, B) we can associate a unique sequence of length n where each element of the sequence corresponds to an element of the set S and is one of $A, B - A$ or $S - B$. Since the number of such sequences is 3^n that is also the number of ordered pairs (A, B) with the desired property.