Tutorial Sheet 5

August 24, 26 and 27

1. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
   (a) How many socks must he take out to be sure that he has at least two socks of the same color?
   (b) How many socks must he take out to be sure that he has at least two black socks?
2. [*] Let $(x_i, y_i), i = 1, 2, 3, 4, 5$ be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.
3. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \mod 5 = a_2 \mod 5$ and $b_1 \mod 5 = b_2 \mod 5$?
4. [*] Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of whose neighbors are boys.
5. [**] Suppose that 21 girls and 21 boys enter a mathematics competition. Furthermore, suppose that each entrant solves at most six questions, and for every boy-girl pair, there is at least one question that they both solved. Show that there is a question that was solved by at least three girls and at least three boys.
6. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends. Use this to show that among any group of 20 people (where any two people are either friends or enemies), there are either four mutual friends or four mutual enemies.
7. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Justify your answer.
8. [*] An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 p.m., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches. Is the statement true if 24 is replaced by
   (a) 2?
   (b) 23?
   (c) 25?
   (d) 30?
9. [*] Let x be an irrational number. Show that for some positive integer j not exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.

Solutions to Tutorial Sheet 5

Solution 1

(a) 3 socks suffice.

(b) 14 are sufficient since at most 12 of these are brown and so the remaining 2 are black.

Solution 2

we consider four holes labeled (0,0), (0,1), (1,0), (1,1). A point (x_i, y_i) is added to the hole labeled $(x_i mod 2, y_i mod 2)$. Since we have four holes and five points/pigeons, two points, say i, j, will be in the same hole. The midpoint of the line joining points i, j has coordinates $((x_i + x_j)/2, (y_i + y_j)/2)$ which are integer since $x_i mod 2 = x_j mod 2$ and $y_i mod 2 = y_j mod 2$.

Solution 4

for contradiction assume that there is a way of seating boys and girls around the table such that no person is seated between two boys. when a set of boys sit together we call it a group. Note that if a group has size 3 or more then we would have a boy sitting between two boys. Hence each group has size 1 or 2 and so the number of groups is at least 13. The chairs between the groups (lets call them gaps) are occupied by girls. Since there are 13 gaps and 25 girls, there is a gap with only one girl in it. This girl then sits between two boys.

Solution 5

Suppose question i is solved by b boys and g girls. Define the type of question i, $t_i = min(a,b)$ and the width of question i, $w_i = max(a,b)$. We have to show that there is a question of type 3. For contradiction assume that all questions are of types 0,1 and 2. Question i is said to cover $t_i \times w_i$ boy-girl pairs. Since $t_i \leq 2$, question i covers at most $2w_i$ pairs. Note that all questions put together should cover all possible boy-girl pairs. Since there are 21×21 boy-girl pairs we have $\sum_i 2w_i \geq 441$ which implies $\sum_i w_i \geq 221$.

If question$i$ is solved by more boys than girls then we call it a boy-question, else we call it a girl question. It is no loss of generality to assume that the total width of boy-questions is $\lceil \frac{221}{2} \rceil = 111$. This implies that some boy, say b, answers at least $\lceil \frac{111}{21} \rceil = 6$ boy-questions. Recall that each boy-question is answered by at most 2 girls and each student answers at most 6 questions. Since$b$ does not solve any other question, there are at most 12 other girls with which$b$ shares a question. This implies there are 9 girls who do not solve the same questions as b. This yields a contradiction.

Solution 6

Recall that in any group of 6 people there are either 3 mutual friends or 3 mutual enemies. In the group of 10 people consider any person x.

1. $x$is either friends with 6 people or enemies with 4 people.

   (a) Amongst the 6 friends of x either there are 3 mutual friends or 3 mutual enemies. Hence including x there are 4 mutual friends or 3 mutual enemies.

   (b) Amongst the 4 enemies of x, if two people are mutual enemies then including x we have 3 mutual enemies. Else the 4 enemies of x are mutual friends.

   Hence in either case there are 4 mutual friends or 3 mutual enemies.

2. $x$is either enemies with 6 people or friends with 4 people. Note that we have just exchanged the role of friends and enemies and so as above we can argue that there are 4 mutual enemies or 3 mutual friends.

Hence we conclude that in any group of 10 people there are either (4 mutual friends or 3 mutual enemies) and (3 mutual friends or 4 mutual enemies).

Now consider a group of 20 people and a person x. Amongst the remaining 19 people x is either friends with 10 people or enemies with 10 people.

1. Amongst the 10 people x is friends with there are 3 mutual friends or 4 mutual enemies. Together with x this implies there are 4 mutual friends or 4 mutual enemies.

2. Amongst the 10 people x is enemies with there are 4 mutual friends or 3 mutual enemies. Together with x this implies there are 4 mutual friends or 4 mutual enemies.

In either case there are 4 mutual friends or 4 mutual enemies.

Solution 7

Let the computers be labeled $c_1, c_2, \ldots, c_8$ and the printers be labeled p_1, p_2, p_3, p_4.Suppose a printer was connected to 4 or less computers. If we chose the 4 computers that this printer was not connected to then none of them would be able to access this printer. This implies that each printer should be connected to at least 5 computers and so we need at least 20 cables.

A solution with 20 cables is as follows. For $1 \leq i \leq 4$, connect c_i to p_i. Further, connect each printer to all four computers in $\{c_5, c_6, c_7, c_8\}$. From the 4 computers chosen, some would be in the set $\{c_1, c_2, c_3, c_4\}$, these would access the coresponding printers. Since the remaining chosen computers can access any of the 4 printers we can ensure that each chosen computer accesses a different printer.

Solution 8

This can be done like the problem solved in class. Define a_i to be the number of matches played by the wrestler in hours 1 to i. Since he plays at least 1 match every hour we have $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$. Define $b_i = a_i + 24$. Hence $25 \leq b_1 < b_2 < \cdots < b_75 \leq 149$. Consider the a_i's and the b_i's as pigeons - there are 150 of these. The holes are numbered 1 through 149 and a pigeon goes to the hole coresponding to its value. By the pigeon hole principle two pigeons will end up in the same hole. These pigeons cannot be two a_i's or two b_i's (since the a_i's and the b_i's are distinct). Hence it must be the case that $b_i = a_i + 24 = a_j$. This implies that in the hours $i+1$ to j the wrestler played exactly 24 matches.

The above argument would continue to work when 24 is replaced by 2 (127 holes, 150 pigeons) or by 23 (148 holes, 150 pigeons).

When replaced by 25 we would have 150 pigeons (thea_i's and b_i's) and 150 holes (the values 1 to 150). If 2 pigeons are in the same hole we would be done as in the arguemnt above. so we consider the case when each hole has exactly one pigeon in it. Since $b_1 \geq 26$, the holes 1 to 25 are occupied by the a_i's. In particular, for $1 \leq i \leq 25$, a_i is in hole i. Hence $a_{25} = 25$ which implies that in the first 25 hours the wrestler plays exactly 25 games.

The solution to part (d) will be added soon.

Solution 9

Consider the n irrational numbers $x, 2x, \ldots, nx$ as the pigeons. For $1 \leq i \leq n$ we define the i^{th} hole as the interval $(\frac{i-1}{n}, \frac{i}{n})$. A pigeon jx goes to the i^{th} hole iff $(i-1)/n < jx - \lfloor jx \rfloor < i/n$. Note that $jx - \lfloor jx \rfloor$ is an irrational number in (0,1) and so it will be assigned to exactly one hole.

If jx goes to the first or the n^{th} hole we are done as the nearest integer to jx is atmost 1/n away. So we assume that the n pigeons go to the $n-2$ holes numbered 2 to $n-1$. This implies that two pigeons, say ix, jx, end up in the same hole. Suppose $i \geq j$. Then $-1/n < (ix - \lfloor ix \rfloor) - (jx - \lfloor jx \rfloor) < 1/n$. Note that $(\lfloor ix \rfloor - \lfloor jx \rfloor)$ is an integer, say k. Further, $1 \leq (i-j) < n$. This implies $k - 1/n < (i-j)x < k + 1/n$ and so the $(i-j)^{th}$ multiple of x differs from an integer by at most $1/n$.