Tutorial Sheet 4

Aug 17, 19, 20

1. Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one mapping from the set to the set of positive integers.
   (a) all positive rational numbers that cannot be written with denominators less than 4
   (b) the real numbers not containing 0 in their decimal representation
   (c) the real numbers containing only a finite number of 1s in their decimal representation
   (d) integers divisible by 5 but not by 7
   (e) the real numbers with decimal representations consisting of all 1s
   (f) the real numbers with decimal representations of all 1s or 9s.
2. Show that if A is an infinite set, then it contains a countably infinite subset.
3. Show that there is no infinite set A such that $|A| < |Z^+| = \aleph_0$.
4. Show that the union of a countable number of countable sets is countable.
5. The Schroeder-Bernstein theorem states that if there is an injective mapping from A to B and an injective mapping from B to A then there exists a bijective mapping from A to B. Use this to argue that
   (a) $(0,1)$ and $[0,1]$ have the same cardinality
   (b) $(0,1)$ and $\Re$ have the same cardinaility
6. Show that there is no one-to-one correspondence from the set of positive integers to the power set of the set of positive integers. [Hint: Assume that there is such a one-to- one correspondence. Represent a subset of the set of positive integers as an infinite bit string with i^{th} bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with itsi^{th} bit equal to the complement of the i^{th} bit of the i^{th} string in the list. Show that this new bit string cannot appear in the list.]
7. Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set of real numbers between 0 and 1. Use this to conclude that $\aleph_0 < |P(Z^+)| = |\Re|$.
8. Show that if S is a set, then there does not exist an onto function f from S to$P(S)$, the power set of S. Conclude that $|S| < |P(S)|$. This result is known as Cantor's theorem. [Hint: Suppose such a function f existed. Let$T = s \in S | s \notin f(s)$ and show that no element s can exist for which $f(s) = T$.]

Solutions to Tutorial Sheet 4

Solution 1

If A is countable then there exists an injective mapping $f : A \rightarrow Z^+$which, for any$S \subseteq A$ gives an injective mapping $g : S \rightarrow Z^+$thereby establishing that S is countable. The contrapositive of this is: if a set is not countable then any superset is not countable.

(a) The rational numbers are countable (done in class) and this is a subset of the rationals. Hence this set is also countable.

(b) this set is not countable. For contradiction suppose the elements of this set in (0,1) are enumerable. As in the diagonalization argument done in class we construct a number, r, in (0,1) whose decimal representation has as its i^{th} digit (after the decimal) a digit different from the i^{th} digit (after the decimal) of the i^{th} number in the enumertation. Note that r can be constructed so that it does not have a 0 in its representaton. Further, by construction r is different from all the other numbers in theeneumareation thus yielding a contradiction.

Solution 2

In class we defined an infinite set as a set that is not finite. Let $S = \{a_1, a_2, \ldots, a_i, \ldots\}$ be a set where $a_i \in A \backslash \{a_1, a_2, \ldots, a_{i-1}\}$.

subset Every element of S is an element of A and hence $S \subseteq A$.

infinite For contradiction assume that S is finite and $S = \{a_1, a_2, \ldots, a_k\}$. Since we could not find an element in $A \backslash \{a_1, a_2, \ldots, a_k\}$ this implies that A is finite which is false.

countable The mapping $f : S \to Z^+$ defined by $f(a_i) = i$, is injective and establishes that S is a countable set. f is also an onto mapping from S to Z^+ since for any positive integer i, there exists $a_i \in S$ such that $f(a_i) = i$. Hence f is a bijection.

Solution 3

Since A is an infinite set, from Q2 it follows that there exists $S \subseteq A$ and a bijection from S to Z^+. Hence $|A| \geq |S| = |Z^+|$.

Solution 4

Let U be a universe and $\mathcal{C} \subseteq 2^U$ the set of all countable subsets of U. Let $\mathcal{A} \subseteq \mathcal{C}$ be a countable subset of $\mathcal{C}$ and $g : \mathcal{A} \to Z^+$ be an injective mapping. Let $A_i \in \mathcal{A}$ be such that $g(A_i) = i$. Since A_i is countable there exists an injective mapping $h_i : A_i \to Z^+$.

Let $X \subseteq U$ be the union of the sets in$\mathcal{A}$. To show X is countable we will show an injective mapping $r : X \to Z^+ \times Z^+$. Since in class we have seen an injective mapping from $Z^+ \times Z^+$ to Z^+ the composition of these two mappings gives us an injective mapping from X to Z^+.

Consider $x \in X$ and amongst all sets in$\mathcal{A}$ containing x, let A_i be the one with smallest index (this set exists by the well-ordering principle). Let $h_i(x) = j$. Then we define $r(x) = (i, j)$. To show this is injective suppose, for contradiction, that $r(x) = r(y) = (i, j)$. Then $x, y \in A_i$ and $h_i(x) = h_i(y) = j$. This contradicts the fact that h_i is injective.

Solution 5(a)

The identity mapping ($h(x) = x$) is an injective mapping from (0,1) to [0,1]. The mapping $h(x) = (2x + 1)/4$ is a bijection from [0,1] to [1/4, 3/4] and hence an injection from [0,1] to (0,1).

Solution 5(b)

Consider the mapping defined as follows:

$$h(x) = \begin{cases} \frac{1}{4x} + \frac{3}{4}, & x > 1 \\ \frac{x}{4} + \frac{1}{2}, & -1 \leq x \leq 1 \\ \frac{1}{4x} + \frac{1}{4}, & x < -1 \end{cases}$$

h is an injective mapping from $\Re$to (0,1). The identity mapping is an injective mapping from (0,1) to $\Re$.

Solution 6

Assume that $f : Z^+ \to 2^{Z^+}$ is a bijection. Let $T \subseteq Z^+$ be defined as: $i \in T$ iff $i \notin f(i)$. Since$f$ is a bijection, f^{-1} exists and let $f^{-1}(T) = j$. Does $j \in T$? By definition of T, $j \in T$ iff $j \notin f(j) = T$. Thus we arrive at a contradiction and hence f does not exist.

Solution 7

A subset of the set of positive integers can be thought of as an infinite bit string with the i^{th} bit 1 if i belongs to the subset and 0 otherwise. With any such bit string we can associate$r \in (0, 1)$ by attaching a decimal point at the beginning of the string and viewing it as the binary expansion of r. Conversely, given a $r \in (0, 1)$ we write the binary expansion of r and make it an infinite bit string by removing the decimal point and appending an infinite sequence of 0's. This infinite bit sring corresponds to a set of positive integers. This gives a bijection between the set $(0, 1)$ and $P(Z^+)$. In question 6 we argued that $|P(Z^+)| > |Z^+|$ and in question 5(b) we showed that there is a bijection between $(0, 1)$ and $\Re$. Hence $\aleph_0 = |Z^+| < |P(Z^+)| = |\Re|$.

Solution 8

Note that in Question 6 we proved this for the case when S is Z^+. The same proof works with S replacing Z^+.