Tutorial Sheet 3

Aug 10, 12, 13

1. Write the numbers 1,2,...,2n on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write |j - k| on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

2. Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [Hint: Work backward, assuming that you did end up with nine zeros.]

3. Prove or disprove that there is a rational number x and an irrational number y such that xy is irrational.

4. Prove that between every two rational numbers there is an irrational number.

5. Let $a_1, a_2, ..., a_n$ be positive real numbers. The arithmetic mean of these numbers is defined by $A = (a_1 + a_2 + \cdots + a_n)/n$, and the geometric mean of these numbers is defined by $G = (a_1 a_2 \ldots a_n)^{1/n}$. Use mathematical induction to prove that $A \geq G$.

6. Show that it is possible to arrange the numbers $1, 2, ..., n$ in a row so that the average of any two of these numbers never appears between them. [Hint: Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

7. Suppose that we want to prove that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$ for all positive integers n.

   (a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

   (b) Show that mathematical induction can be used to prove the stronger inequality $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$ for all integers greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

8. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

Solutions to Tutorial Sheet 3

Solution 1

we will prove that the invariant "there are an odd number of odd numbers on the board" holds througout the process.

1. it is true initially since there are n odd numbers between 1 and 2n and n is odd.

2. when both j and k are odd then $|j - k|$ is even and so the number of odd numbers decreases by 2. When both j and k are even then $|j - k|$ is even and so the number of odd numbers remain unchanged. When one of j, k is even and the other odd then $|j - k|$ is odd and so the number of odd numbers remains unchanged. Thus the number of odd numbers either remains unchanged or decreases by 2. Hence the invariant holds after each step.

3. When only one number remains, the invariant holds and so this number must be odd.

We could also have used other invariants like "the sum of the numbers on the board is odd".

Solution 2

We will prove that the invariant -there is at least one "1" and one "0"- holds throughout the process. The invariant holds initially. Suppose the invariant holds at round i. Since not all bits are identical there would be two adjacent bits which would be different and these would lead to a 1 bit in round $i + 1$. The number of 0 and 1 bits cannot be equal (there are 9 bits in all) and hence in round$i$ there will be two adjacent bits that are identical. This will generate a 0 in round $i + 1$. Hence the invariant holds in round $i + 1$.

Solution 3

Let x be 1 and y be $\sqrt{2}$.Then $xy = \sqrt{2}$and is irrational.

Solution 4

Let $\frac{a}{b}, \frac{c}{d}$ be two arbitrary rational numbers. Let $r = lcm(b, d)$ and $\frac{p}{r} = \frac{a}{b}$ and $\frac{q}{r} = \frac{c}{d}$ and $p < q$. Since $0 < 1/\sqrt{2}$ <1, we have $\frac{a}{b} < \frac{p+1/\sqrt{2}}{r} < \frac{c}{d}$. Suppose $\frac{p+1/\sqrt{2}}{r}$ is rational and equals$\frac{x}{y}$. Then $\sqrt{2} = \frac{y}{rx-py}$ which implies that $\sqrt{2}$ is rational. This is a contradiction and hence $\frac{p+1/\sqrt{2}}{r}$ is irrational.

Solution 5

Let $A(S), G(S)$ be the arithmetic and geometric means of a set of numbers S. We will prove that $\forall S : A(S) \geq G(S)$ by induction on the cardinalty of S.

1. For $|S| = 2$ we have to prove that $x_1 + x_2 \geq 2(x_1 x_2)^{1/2}$, where $S = \{x_1, x_2\}$. Since $(x_1 - x_2)^2 \geq 0$, we have $(x_1 + x_2)^2 \geq 4x_1x_2$. Taking squareroots we obtain the desired claim.

2. Our induction hypothesis is that $\forall S, \quad 2 \leq |S| < n, \quad A(S) \geq G(S)$ and we have to prove that $A(S) \geq G(S)$ for a set $S = \{a_1, a_2, \ldots, a_n\}$. We first consider the case when n is odd. Let $a = A(S)$ and $S' = S + \{a\}$. Let $S_1 = \{a_1, a_2, \ldots, a_{(n+1)/2}\}$ and $S_2 = S' \S_1$. Let $x_1 = A(S_1), x_2 = A(S_2), y_1 = G(S_1), y_2 = G(S_2)$. Since$|S_1| = |S_2| = \frac{n+1}{2}$ and $\frac{n+1}{2} < n$ we can apply the induction hypothesis on S_1, S_2 to get $x_1 \geq y_1$ and $x_2 \geq y_2$. Now $A(S') = \frac{x_1+x_2}{2} \geq \frac{y_1+y_2}{2} \geq (y_1y_2)^{1/2} = G(S')$. Hence $A(S')^{n+1} \geq G(S')^{n+1} = G(S)^n \cdot a = G(S)^n \cdot A(S)$. Since $A(S) = A(S')$, the above implies $A(S) \geq G(S)$. When n is even then we define $S' = S$ and the argument remains the same.

Solution 6

We want to prove that for every$n$ there exists a permutation$\pi_n$of the numbers $\{1, 2, \ldots, n\}$ such that the average of two numbers does not lie between them. If $(\pi_n(1), \pi_n(2), \ldots, \pi_n(n))$ is the ordering then $\forall i, j, k, \quad i < j < k, \rightarrow \pi_n(j) \neq (\pi_n(i) + \pi_n(k))/2$. Our induction hypothesis is that for all $k < n$ such a permutation π_k exists. For $k = 2$ the ordering would be (1,2). We partition the numbers from $\{1, 2, \ldots, n\}$ into odd and even numbers. Since the average of an odd and an even number is not an integer we can order the set $\{1, 2, \ldots, n\}$ by putting all the odd numbers before the even ones. However both these sets should be arranged such that the average of any two numbers does not lie between them. Let $o = \lceil \frac{n}{2} \rceil$, $e = \lfloor \frac{n}{2} \rfloor$ be the sizes of the odd and even sets. Then $\pi_n(i) = 2\pi_o(i) - 1$ if $i \leq o$, $\pi_n(i) = 2\pi_e(i - o)$ if $i > o$.

We now need to argue that for any $i < j < k$, $\pi_n(j) \neq (\pi_n(i) + \pi_n(k))/2$. For contradiction assume that there exist such i, j, k.

1. If $i \leq o < k$ then in permutation π_n the number at position i is odd while the number at position k is even. Since their average is not an integer the number at position j will be different.

2. Let $i < j < k \leq o$. Then $\pi_n(j) = (\pi_n(i) + \pi_n(k))/2$ implies $\pi_o(j) = (\pi_o(i) + \pi_o(k))/2$. However, by our induction hypothesis π_o has the property that for all $i < j < k \leq o$, $\pi_o(j) \neq (\pi_o(i) + \pi_o(k))/2$.

3. Let $o < i < j < k$. Then $\pi_n(j) = (\pi_n(i) + \pi_n(k))/2$ implies $\pi_e(j - o) = (\pi_e(i - o) + \pi_e(k - o))/2$. However, by our induction hypothesis π_e has the property that for all $o < i' < j' < k'$, $\pi_e(j') \neq (\pi_e(i') + \pi_e(k'))/2$.

Solution 7(a)

Our inductive hypothesis would be that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$ and in the inductive step we wish to prove that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3(n+1)}}$. Using the induction hypothesis the LHS can be bounded by $\frac{1}{\sqrt{3n}} \frac{2n+1}{2n+2}$ and to show that this is atmost $\frac{1}{\sqrt{3(n+1)}}$ we would have to establish that $\frac{2n+1}{2n+2} \leq \frac{\sqrt{n}}{\sqrt{n+1}}$. For n=1, this transaltes to showing $\frac{3}{4} \leq \frac{1}{\sqrt{2}}$ which is not true.

Solution 7(b)

For the base case, $n = 1$, the stronger inequality is $1/2 \leq 1/2$ which is true. When proving this stronger inequality by induction we have to establish that $\frac{2n+1}{2n+2} \leq \frac{\sqrt{3n+1}}{\sqrt{3n+4}}$. This is equivalent to $3.(2n + 1)^2 \leq (3n + 1)(4n + 3)$ which can be shown to be true.

Solution 8

Let $P(n)$ be the proposition that a rectangular chessboard of size $2n \times m$, from which either one black and one white square or no square have been removed, can be covered with dominoes. Assuming $P(n)$ is true for all $n < k$, we want to show that $P(k)$ is true. We divide the chessboard into two boards of size $2 \times m$ and $2(k-1) \times m$ and consider the possible arrangements of the removed black and white sqaures.

1. if both the removed squares lie in one of the two smaller boards then we use the induction hypothesis to cover the two boards with dominoes and this gives a covering of the $2k \times m$ board.

2. if one of the removed square is from the $2 \times m$ board and the other from the $2(k-1) \times m$ board then we remove squares $(2, j)$ and $(3, j)$ where j is chosen such that $(2, j)$ has color different than that of the square removed from the $2 \times m$ board. It follows that the square (3,j) would have a color different from that of the square removed from the $2(k-1) \times m$ board and so we can apply the induction hypothesis on these two smaller boards to cover them with dominoes. We place one domino covering squares $(2, j)$ and $(3, j)$ and this gives a covering of the $2k \times m$ board.