COL111: Discrete Mathematical Structures

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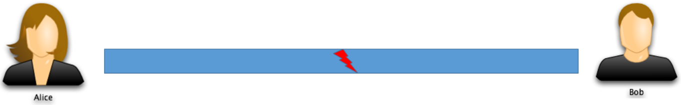
Error Correcting Codes

Problem

Alice wants to send a sequence of integers $m_1, m_2, ..., m_n$ to Bob over a faulty communication channel that may *drop* at most k of the numbers sent by Alice. Assume that $\forall i, x_i \in \{0, 1, ..., q-1\}$ for some integer $q \geq n + k$. Suggest a method for Alice to communicate her message to Bob.



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  - Claim 1: Alice can find the unique univariate polynomial $P(.)$ of degree $(n-1)$ with coefficients in $\{0, 1, ..., q-1\}$ s.t.
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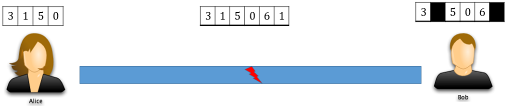

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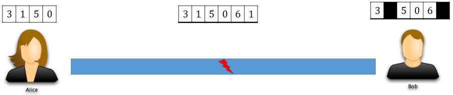

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Polynomials

Properties of Polynomials

- A univariate (one variable) polynomial of degree d is of the form $p(x) = a_d x^{d-1} + a_{d-1} x^{d-1} + ... + a_0$, where x and *coefficients* a_i's are real numbers.
- A real number a is said to be a *root* of a polynomial $p(x)$ iff $p(a) = 0$.

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Proof.

- Find a polynomial $p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0$ such that $p(x_i) = y_i$ for all i.
- This can be done by solving the following system:

$$\begin{pmatrix} x_1^d & x_1^{d-1} & \dots & x_1 & 1 \\ x_2^d & x_2^{d-1} & \dots & x_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d+1}^d & x_{d+1}^{d-1} & \dots & x_{d+1} & 1 \end{pmatrix} \times \begin{pmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{d+1} \end{pmatrix}$$

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- Is the above matrix invertible?

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- Is the above matrix invertible?
  - Yes, since the determinant $(= \prod_{i>j}(x_i - x_j))$ is non-zero as long as $x_1, x_2, ..., x_{d+1}$ are distinct.
- What about uniqueness?
  - Suppose there is another polynomial $q(x) \neq p(x)$ such that $\forall i, q(x_i) = y_i$. But then $r(x) = p(x) - q(x)$ is a degree d polynomial with $d+1$ roots.

$\square$

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- For all i define the polynomial $\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$.
- Claim: The unique degree d polynomial in the above theorem is given by $p(x) = \sum_i y_i \cdot \Delta_i(x)$.

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  - Suppose we are given $(1, 3), (2, 1), (3, 5), (4, 0)$, then the degree 3 polynomial that "fits" these pairs is given by:

$$
\begin{aligned}
p(x) \quad = \quad & \frac{3(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + \frac{1(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + \\
& \frac{5(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + \frac{0(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}
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$$p(x) = \frac{(x-2)(x-3)(x-4)}{-2} + \frac{(x-1)(x-3)(x-4)}{2} + \frac{5(x-1)(x-2)(x-4)}{-2}.$$

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- This method of "fitting" a polynomial is known as *Lagrange interpolation*.

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- Do both the above theorems hold when the variable x and all coefficients are restricted to be rational numbers?

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- Do both the above theorems hold when the variable x and all coefficients are restricted to be rational numbers? Yes.
- Do both the above theorems hold when the variable x and all coefficients are restricted to be integers? No.

Properties of Polynomials

- Let $q > 1$ be some prime number.
- Consider polynomials where the variable x and coefficients can only take values from the set $\{0, ..., q-1\}$.
- All arithmetic operations are performed modulo q.

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- Do the above theorems hold? Yes.
- Where did we use the fact that q is a prime number?
- The set $\{0, 1, ..., q-1\}$ for prime q along with addition and multiplication modulo q is something known as a *Finite Field*. This is useful in a lot of areas of computer science.

- Let $n = 4, k = 2, q = 7$.
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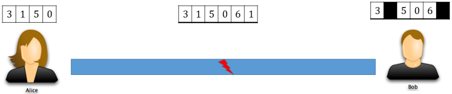

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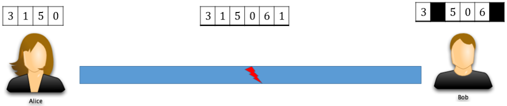

- Use Lagrange interpolation to determine the polynomial that Alice uses.
  - We know $P(1)\ (mod\ 7) = 3, P(2)\ (mod\ 7) = 1, P(3)\ (mod\ 7) = 5, P(4)\ (mod\ 7) = 0$.
  - $Q(x) = \frac{3\cdot(x-2)(x-3)(x-4)}{-6} + \frac{1\cdot(x-1)(x-3)(x-4)}{2} + \frac{5\cdot(x-1)(x-2)(x-4)}{-2}$
  - $Q(x) \equiv 3 \cdot (x-2)(x-3)(x-4) + 4 \cdot (x-1)(x-3)(x-4) + 1 \cdot (x-1)(x-2)(x-4)\ (mod\ 7)$.
  - $Q(x) \equiv (x^3 + 4x^2 + 5)\ (mod\ 7)$.
  - So, $P(x) = x^3 + 4x^2 + 5$.

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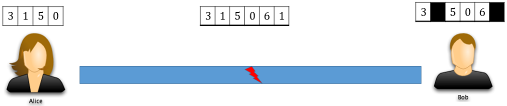

- Use Lagrange interpolation to determine the polynomial that Alice uses.
- Use Lagrange interpolation to determine the polynomial that Bob reconstructs.

Secret Sharing

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- You cannot entrust any single person with this key.
- Ideally, you would want to split this key s into n parts and give each part to a responsible person with the following two properties:
  - If any k (or more) people get together, then they can reconstruct the key s.
  - Less than k people cannot reconstruct s using their shares.

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  - Less than k people cannot reconstruct s using their shares.
- Idea using (finite field) polynomials:
  - Pick a large prime $q >> s, n$.
  - Pick a random polynomial of degree $(k-1)$ such that $P(0) \ (mod \ q) = s$ and give $P(1) \ (mod \ q), P(2) \ (mod \ q), ..., P(n) \ (mod \ q)$ as shares to n people.

End