Partially Observable Markov Decision Processes

Mausam
(slides by Dieter Fox)

Stochastic Planning: MDPs

Static

Environment

Fully Observable

Stochastic

What action next?

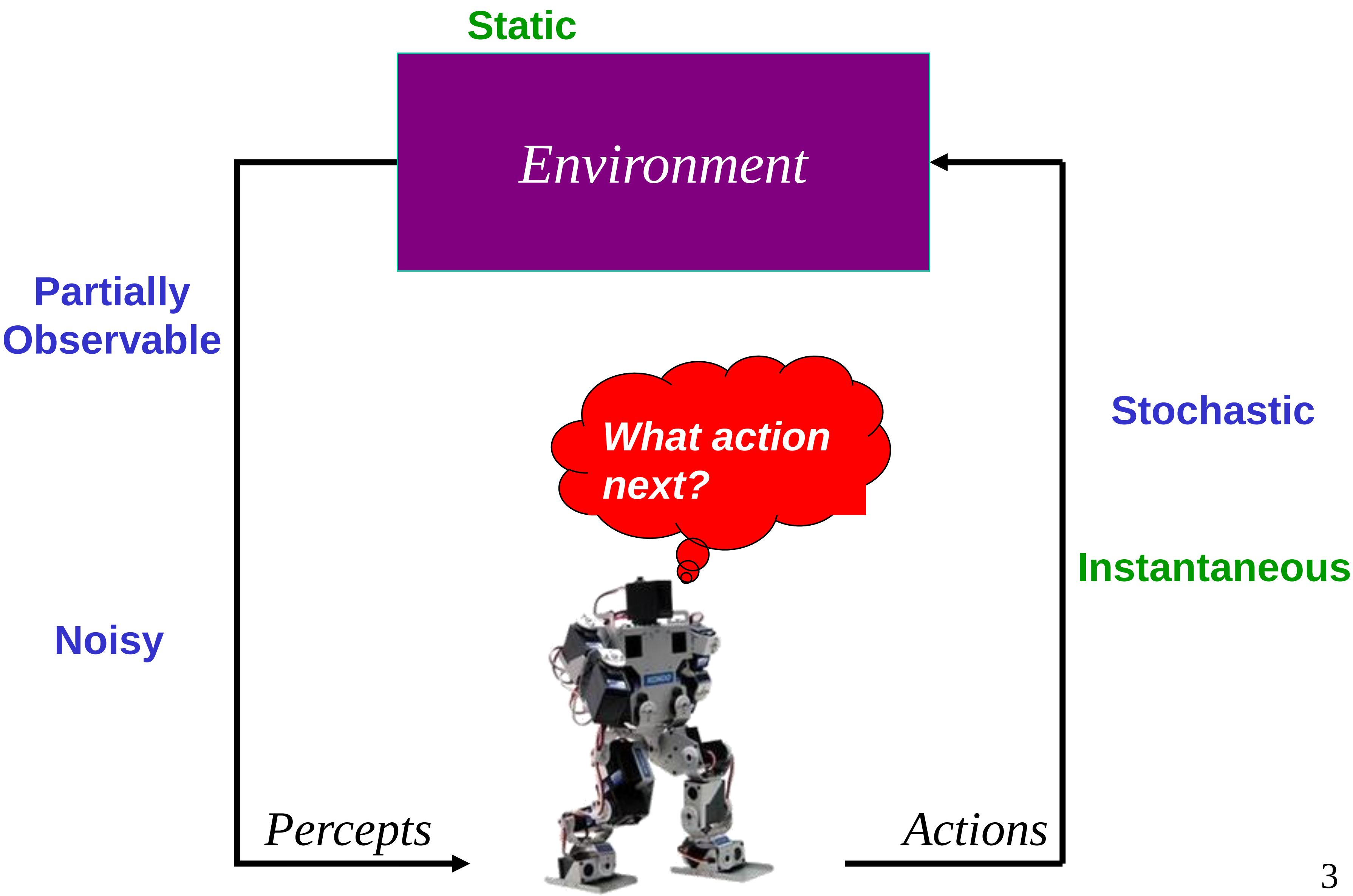
Instantaneous

Perfect

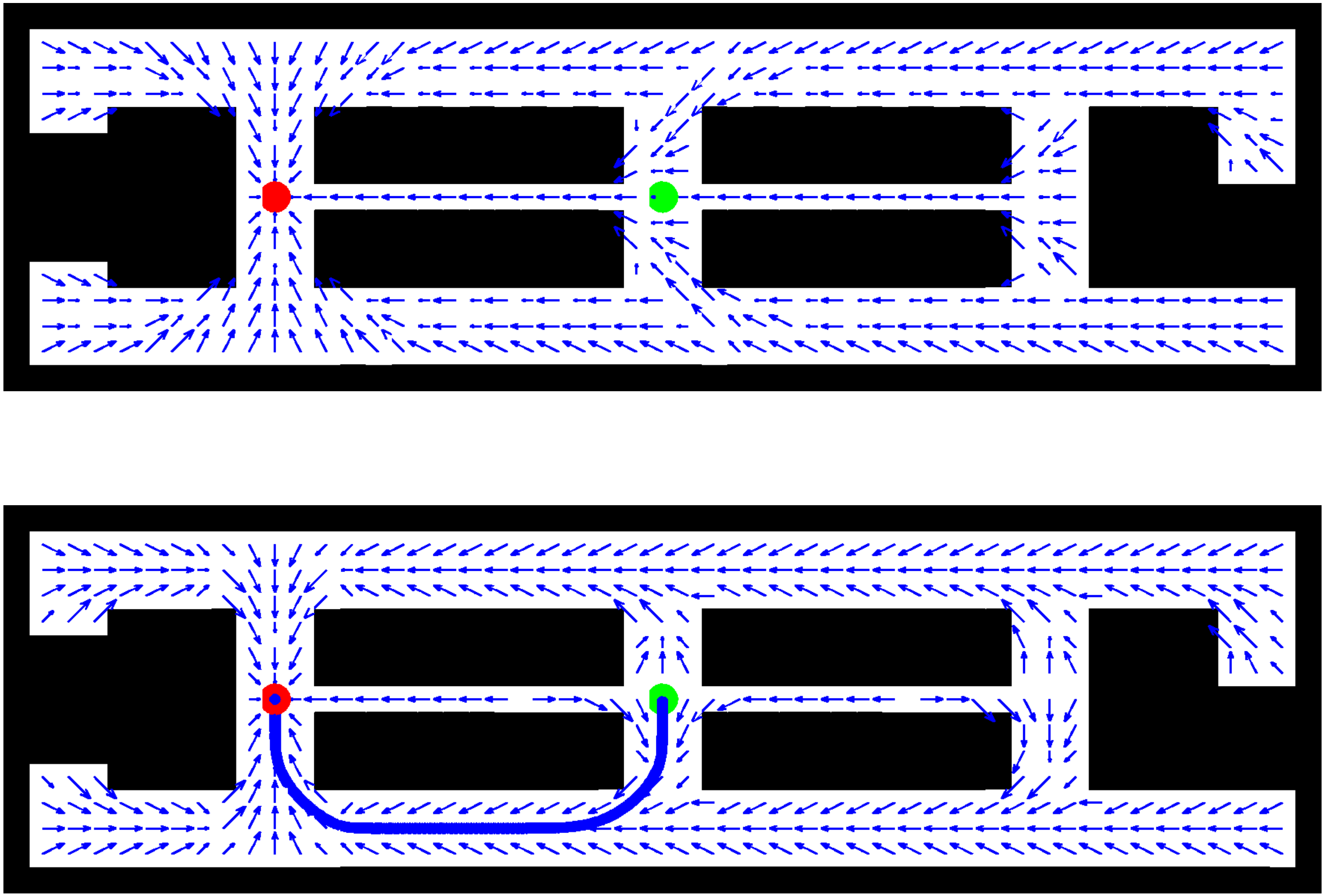
Percepts

Actions

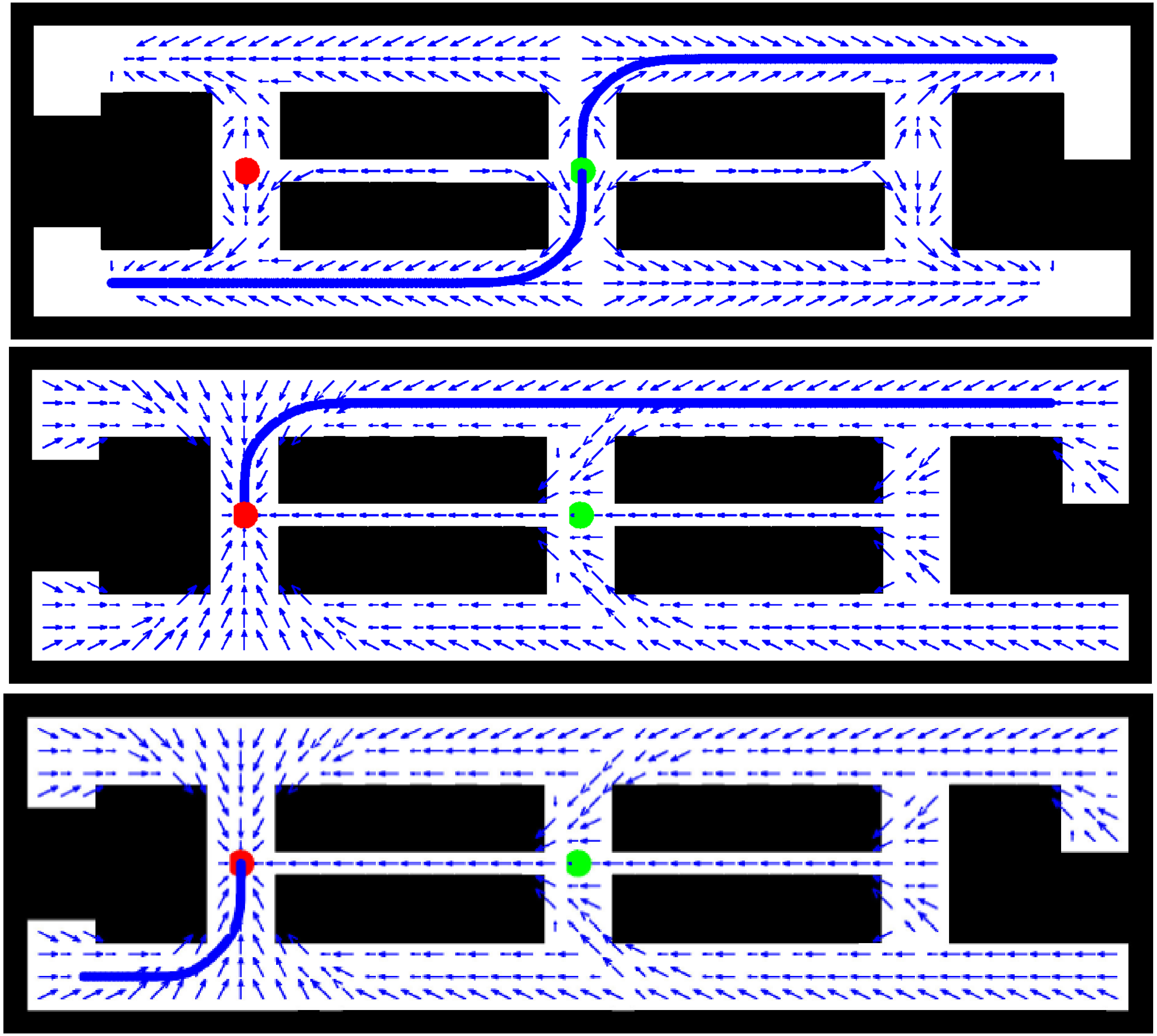
Partially Observable MDPs



Stochastic, Fully Observable



Stochastic, Partially Observable



POMDPs

- In POMDPs we apply the very same idea as in MDPs.

- Since the state is not observable,

  the agent has to make its decisions based on the belief state which is a posterior distribution over states.

- Let b be the belief of the agent about the current state
- POMDPs compute a value function over belief space:

$$V_T(b) = \max_a \left[r(b,a) + \gamma \int V_{T-1}(b') p(b' \mid b, a) \, db' \right]$$

POMDPs

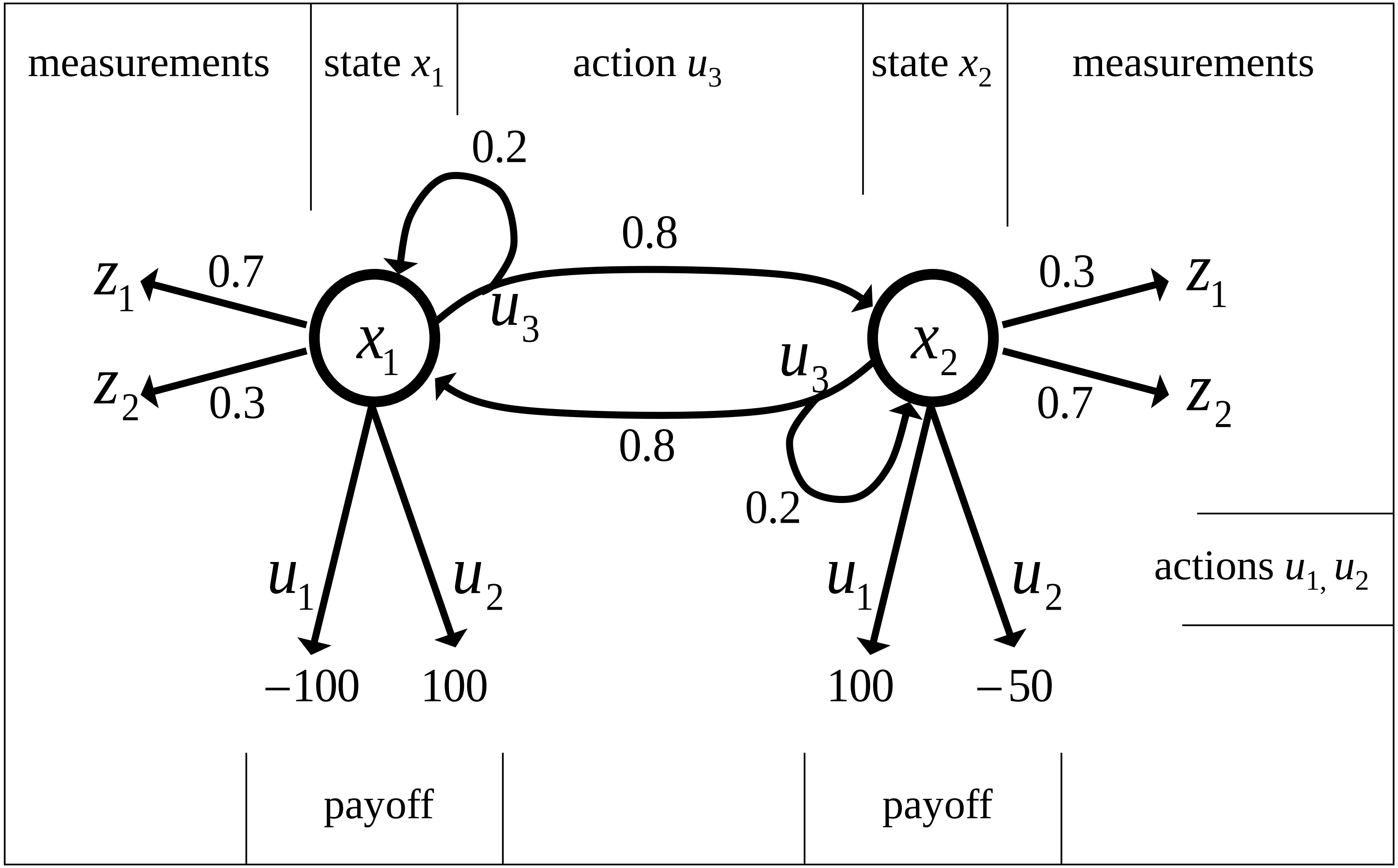
- Each belief is a probability distribution,
  - value fn is a function of an entire probability distribution.
- Problematic, since probability distributions are continuous.
- Also, we have to deal with huge complexity of belief spaces.

- For finite worlds with finite state, action, and observation spaces and finite horizons,
  - we can represent the value functions by piecewise linear functions.

Applications

- Robotic control
  - helicopter maneuvering, autonomous vehicles
  - Mars rover - path planning, oversubscription planning
  - elevator planning
- Game playing - backgammon, tetris, checkers
- Neuroscience
- Computational Finance, Sequential Auctions
- Assisting elderly in simple tasks
- Spoken dialog management
- Communication Networks – switching, routing, flow control
- War planning, evacuation planning

An Illustrative Example



The Parameters of the Example

- The actions u_1 and u_2 are terminal actions.
- The action u_3 is a sensing action that potentially leads to a state transition.
- The horizon is finite and γ=1.

$$
\begin{aligned}
r(x_1, u_1) &= -100 & \quad r(x_2, u_1) &= +100 \\
r(x_1, u_2) &= +100 & \quad r(x_2, u_2) &= -50 \\
r(x_1, u_3) &= -1 & \quad r(x_2, u_3) &= -1
\end{aligned}
$$

$$
\begin{aligned}
p(x'_1 | x_1, u_3) &= 0.2 & \quad p(x'_2 | x_1, u_3) &= 0.8 \\
p(x'_1 | x_2, u_3) &= 0.8 & \quad p(z'_2 | x_2, u_3) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(z_1 | x_1) &= 0.7 & \quad p(z_2 | x_1) &= 0.3 \\
p(z_1 | x_2) &= 0.3 & \quad p(z_2 | x_2) &= 0.7
\end{aligned}
$$

Payoff in POMDPs

- In MDPs, the payoff (or return) depended on the state of the system.
- In POMDPs, however, the true state is not exactly known.
- Therefore, we compute the **expected payoff by integrating over all states**:

$$
\begin{aligned}
r(b,u) &= E_x[r(x,u)] \\
&= \int r(x,u)p(x)\ dx \\
&= p_1\ r(x_1,u) + p_2\ r(x_2,u)
\end{aligned}
$$

Payoffs in Our Example (1)

- If we are totally certain that we are in state x_1 and execute action u_1, we receive a reward of -100
- If, on the other hand, we definitely know that we are in x_2 and execute u_1, the reward is +100.
- In between it is the linear combination of the extreme values weighted by the probabilities

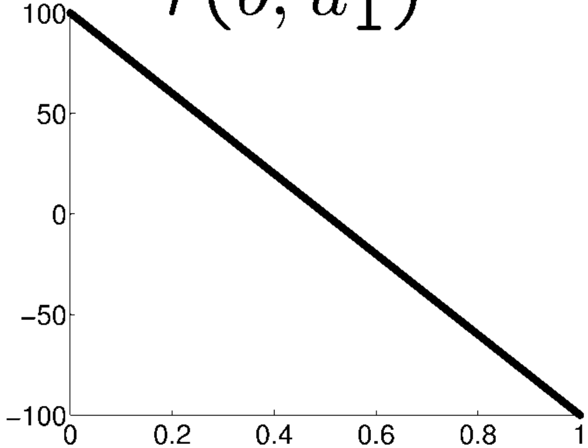
$$\begin{aligned} r(b, u_1) &= -100\, p_1 + 100\, p_2 \\ &= -100\, p_1 + 100\, (1 - p_1) \end{aligned}$$

$$r(b, u_2) = 100\, p_1 - 50\, (1 - p_1)$$

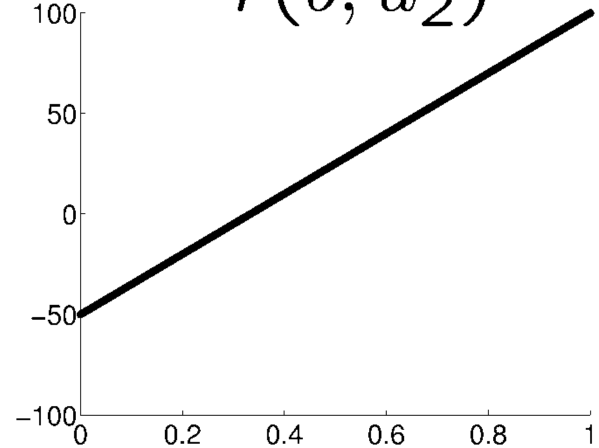
$$r(b, u_3) = -1$$

Payoffs in Our Example (2)

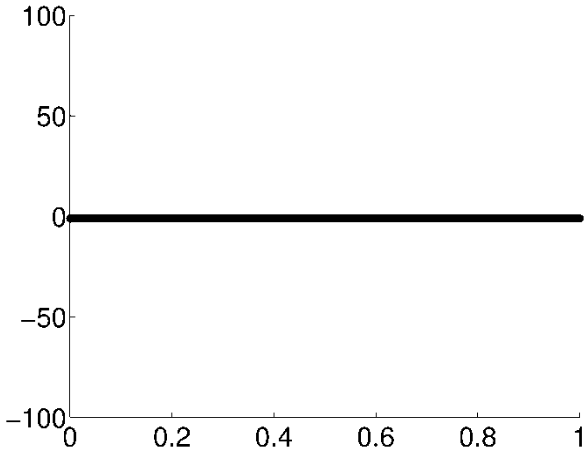
$r(b, u_1)$



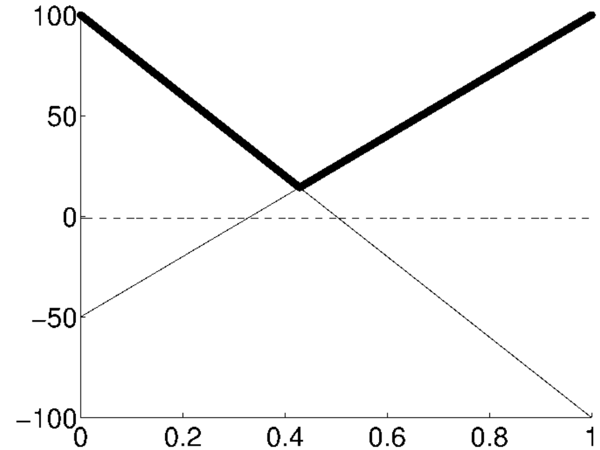
$r(b, u_2)$



$r(b, u_3)$



$V_1(b) = max_u r(b, u)$



The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.
- In our example, the optimal policy for T=1 is

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \leq \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions at each point

$$\begin{aligned} V_1(b) &= \max_u r(b,u) \\ &= \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ -1 & \end{array} \right\} \end{aligned}$$

- It is piecewise linear and convex.

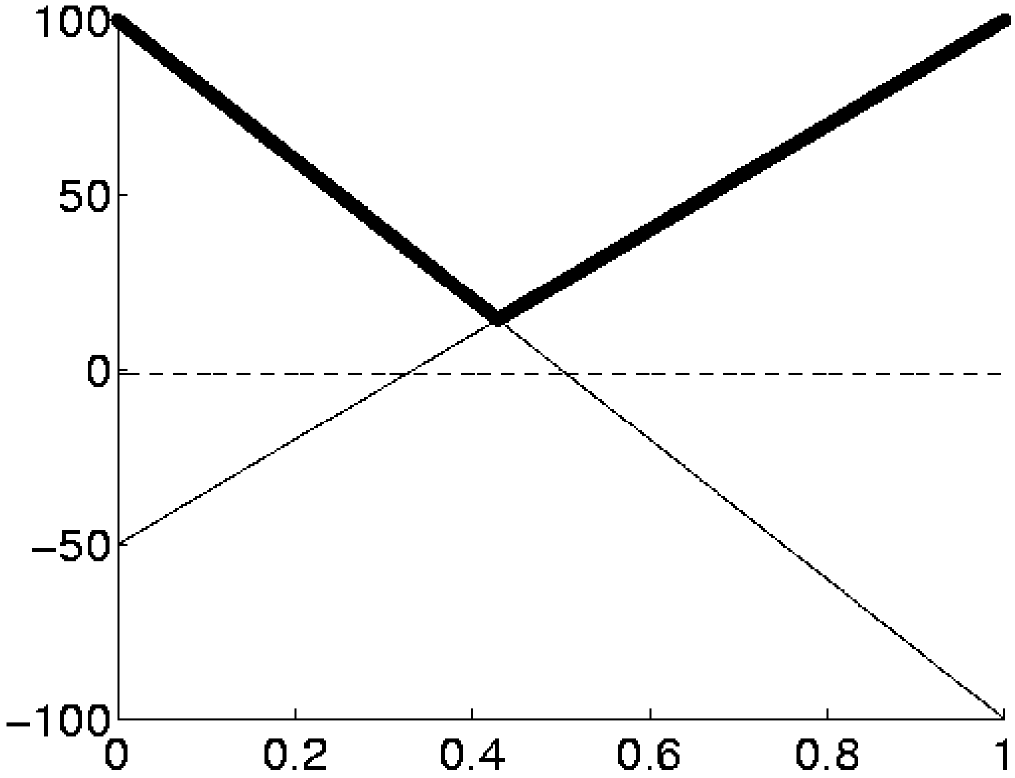
Pruning

- If we carefully consider $V_1(b)$, we see that only the first two components contribute.
- The third component can therefore safely be pruned away from $V_1(b)$.

$$V_1(b) \;=\; \max\left\{\begin{array}{rr} -100\,p_1 & +100\,(1-p_1) \\ 100\,p_1 & -50\,(1-p_1) \end{array}\right\}$$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.



$V_1(b)$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.

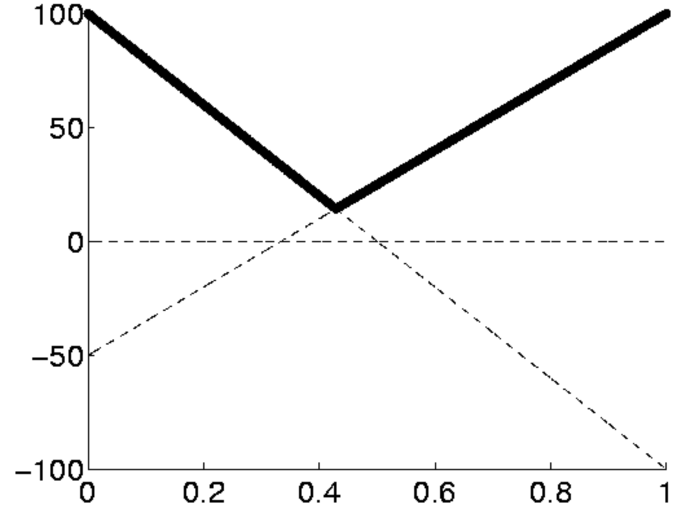
$$p'_1 = \frac{0.7\, p_1}{p(z_1)}$$

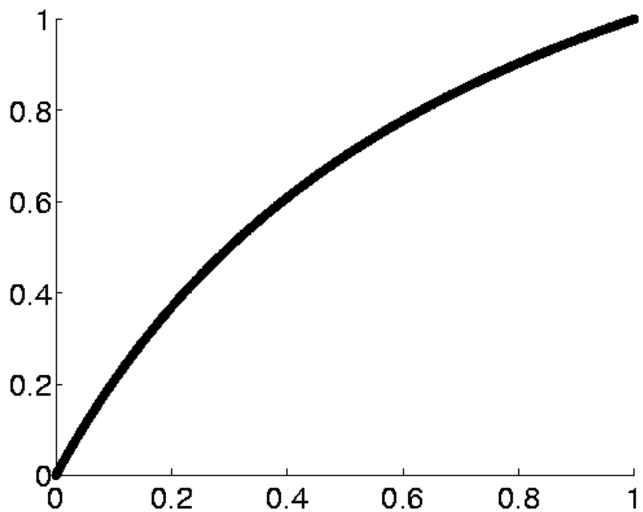
$$p'_2 = \frac{0.3(1-p_1)}{p(z_1)}$$

$$p(z_1) = 0.7\, p_1 + 0.3(1-p_1) = 0.4\, p_1 + 0.3$$

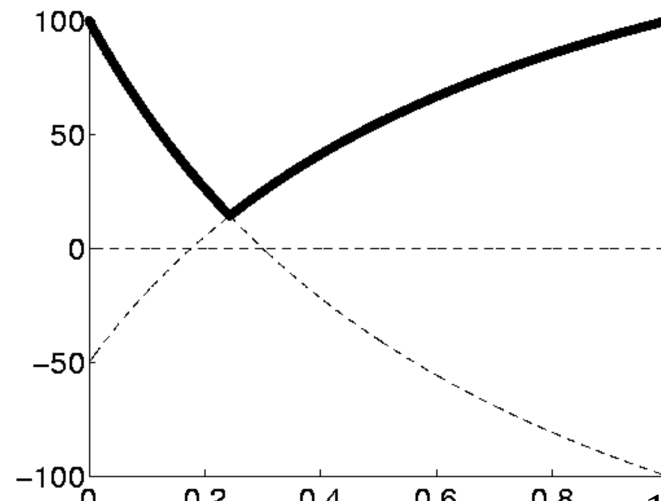
Value Function

$V_1(b)$





$b'(b|z_1)$



$V_1(b|z_1)$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.
- Thus $V_1(b \mid z_1)$ is given by

$$
\begin{aligned}
V_1(b \mid z_1) &= \max \left\{ \begin{array}{cc} -100 \cdot \frac{0.7\, p_1}{p(z_1)} & +100 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \\ \\ 100 \cdot \frac{0.7\, p_1}{p(z_1)} & -50 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \end{array} \right\} \\
&= \frac{1}{p(z_1)} \max \left\{ \begin{array}{cc} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$\bar{V}_1(b) = E_z[V_1(b \mid z)] = \sum_{i=1}^{2} p(z_i) V_1(b \mid z_i)$$

$$= \sum_{i=1}^{2} p(z_i) V_1\left(\frac{p(z_i \mid x_1) p_1}{p(z_i)} \right)$$

$$= \sum_{i=1}^{2} V_1\left(p(z_i \mid x_1) p_1 \right)$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

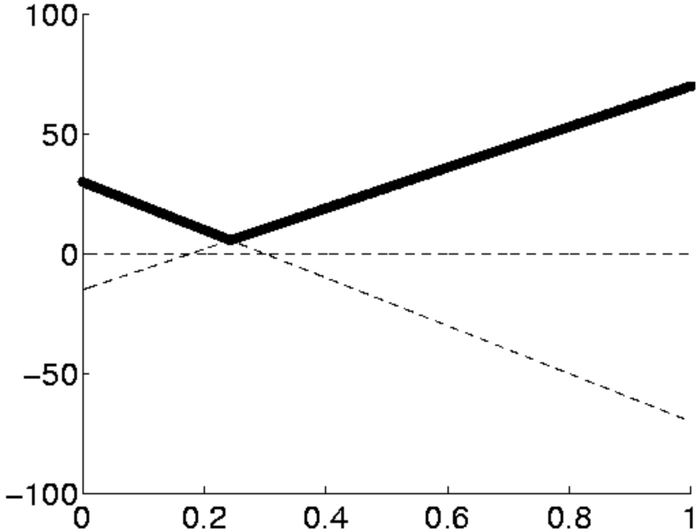
$$
\begin{aligned}
\bar{V}_1(b) &= E_z[V_1(b \mid z)] \\
&= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\
&= \max\left\{ \begin{array}{rr} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\} \\
&\quad + \max\left\{ \begin{array}{rr} -30\, p_1 & +70\,(1-p_1) \\ 30\, p_1 & -35\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

Resulting Value Function

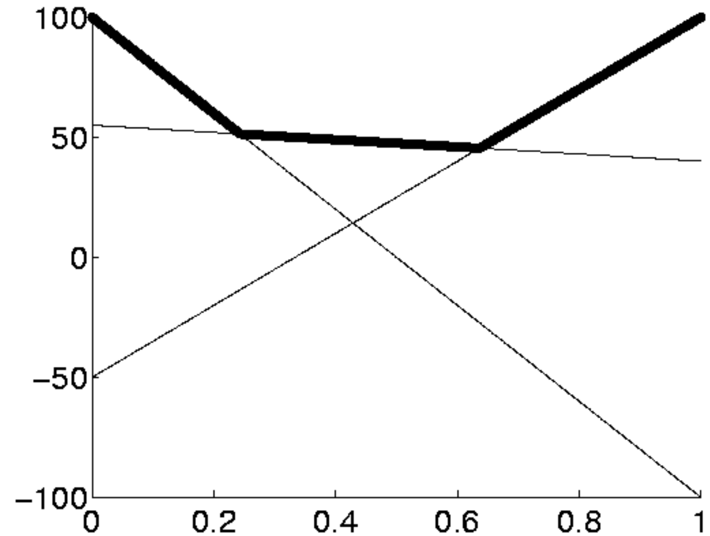
- The four possible combinations yield the following function which then can be simplified and pruned.

$$
\begin{aligned}
\bar{V}_1(b) &= \max \left\{ \begin{array}{llll} -70\, p_1 & +30\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ -70\, p_1 & +30\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \end{array} \right\} \\
&= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ +40\, p_1 & +55\,(1-p_1) \\ +100\, p_1 & -50\,(1-p_1) \end{array} \right\}
\end{aligned}
$$

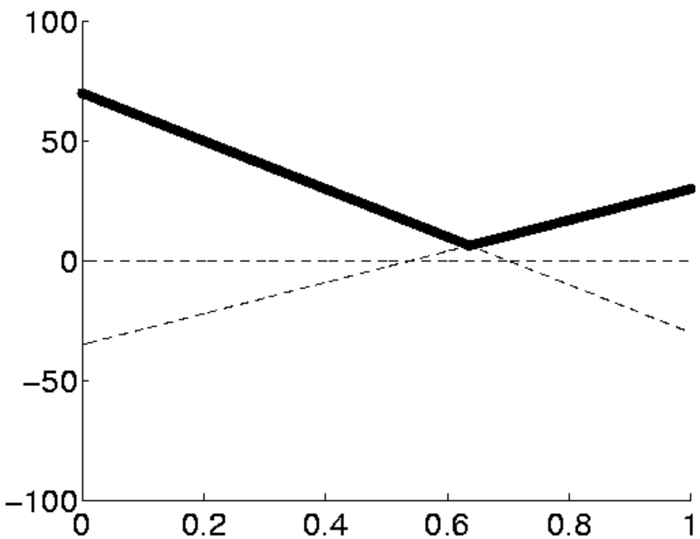
Value Function



$p(z_1)\ V_1(b|z_1)$



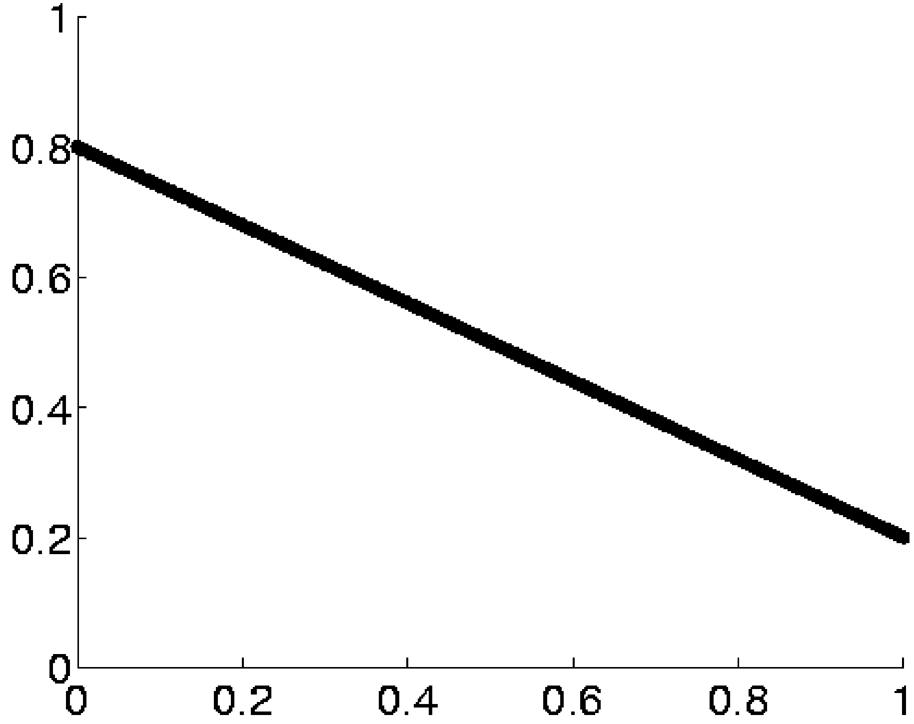
\bar{V}$_1$(b)



$p(z_2)\ V_2(b|z_2)$

State Transitions (Prediction)

- When the agent selects u_3 its state potentially changes.
- When comp[...] have to take these [...] account.



$$
\begin{aligned}
p_1' &= \\
&= \\
&= \\
&= 0.8 - 0.6 p_1
\end{aligned}
$$

Resulting Value Function after executing u_3

- Taking the state transitions into account, we finally obtain.

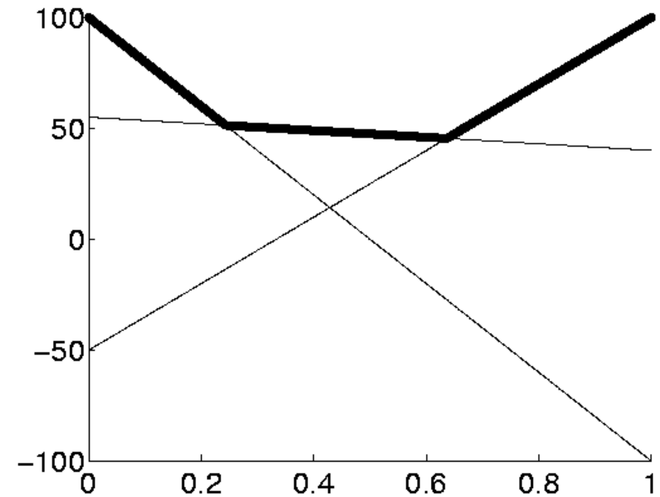
$$\bar{V}_1(b) = \max \left\{ \begin{array}{llll} -70\,p_1 & +30\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ -70\,p_1 & +30\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \end{array} \right\}$$

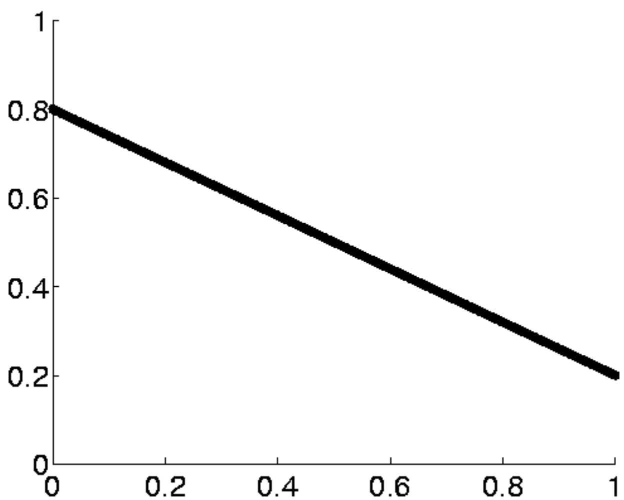
$$= \max \left\{ \begin{array}{ll} -100\,p_1 & +100\,(1-p_1) \\ +40\,p_1 & +55\,(1-p_1) \\ +100\,p_1 & -50\,(1-p_1) \end{array} \right\}$$

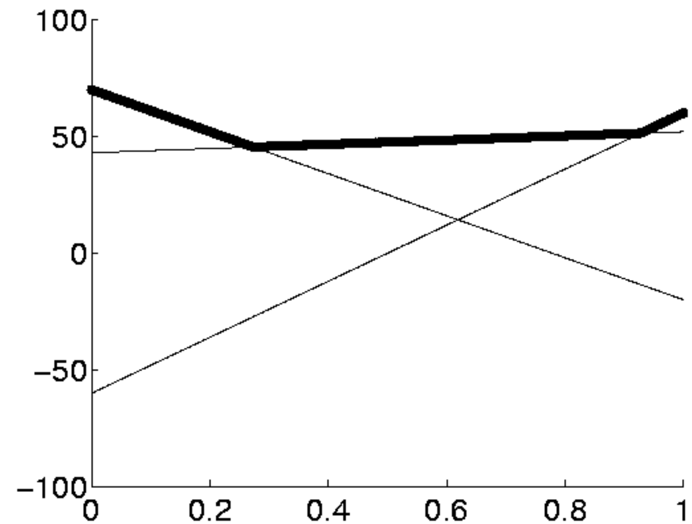
$$\bar{V}_1(b \mid u_3) = \max \left\{ \begin{array}{ll} 60\,p_1 & -60\,(1-p_1) \\ 52\,p_1 & +43\,(1-p_1) \\ -20\,p_1 & +70\,(1-p_1) \end{array} \right\}$$

Value Function after executing u_3

$\bar{V}_1(b)$







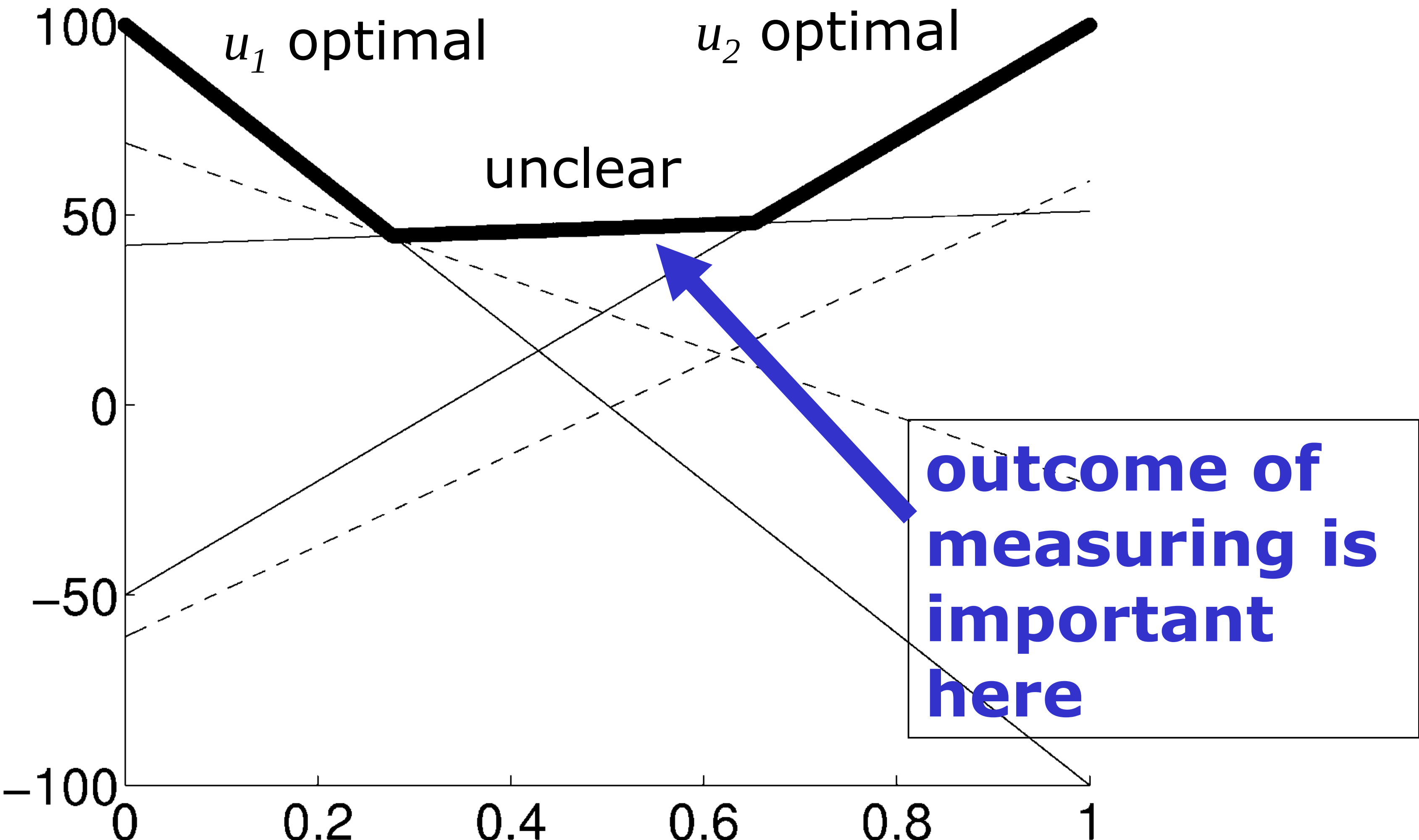
$\bar{V}_1(b|u_3)$

Value Function for T=2

- Taking into account that the agent can either directly perform u_1 or u_2 or first u_3 and then u_1 or u_2, we obtain (after pruning)

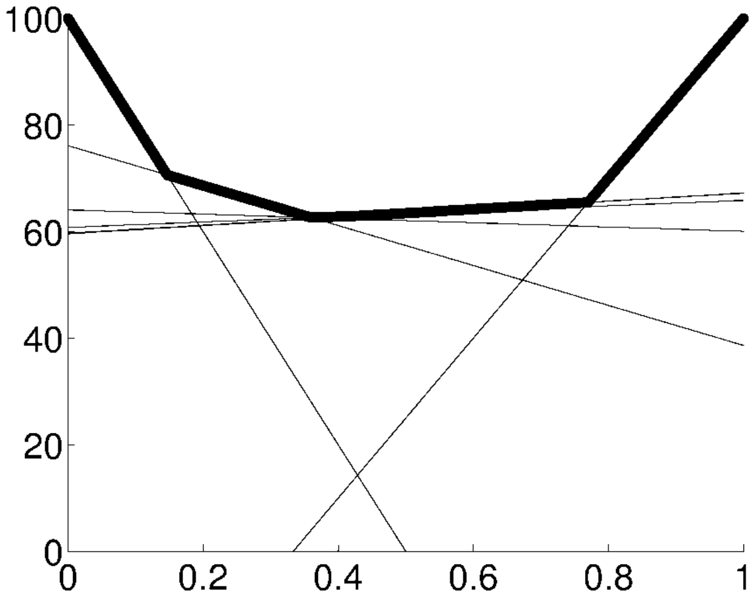
$$\bar{V}_2(b) = \max \left\{ \begin{array}{rr} -100\,p_1 & +100\,(1-p_1) \\ 100\,p_1 & -50\,(1-p_1) \\ 51\,p_1 & +42\,(1-p_1) \end{array} \right\}$$

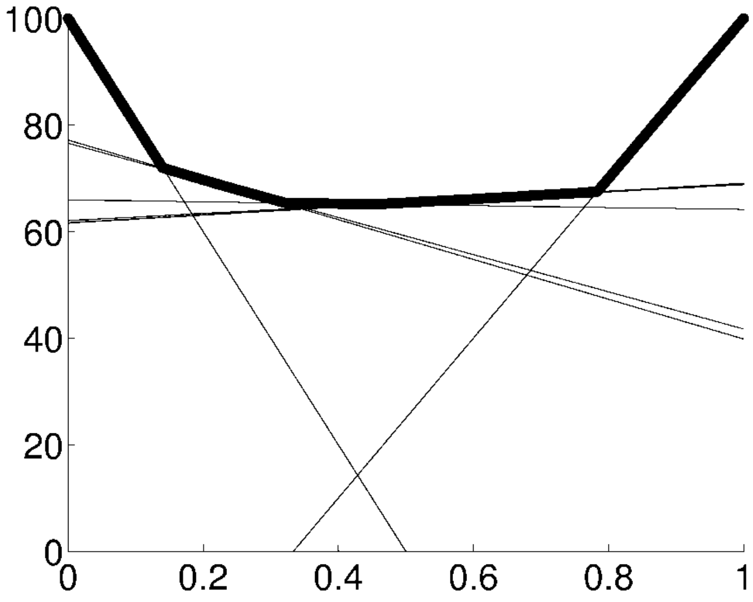
Graphical Representation of $V_2(b)$



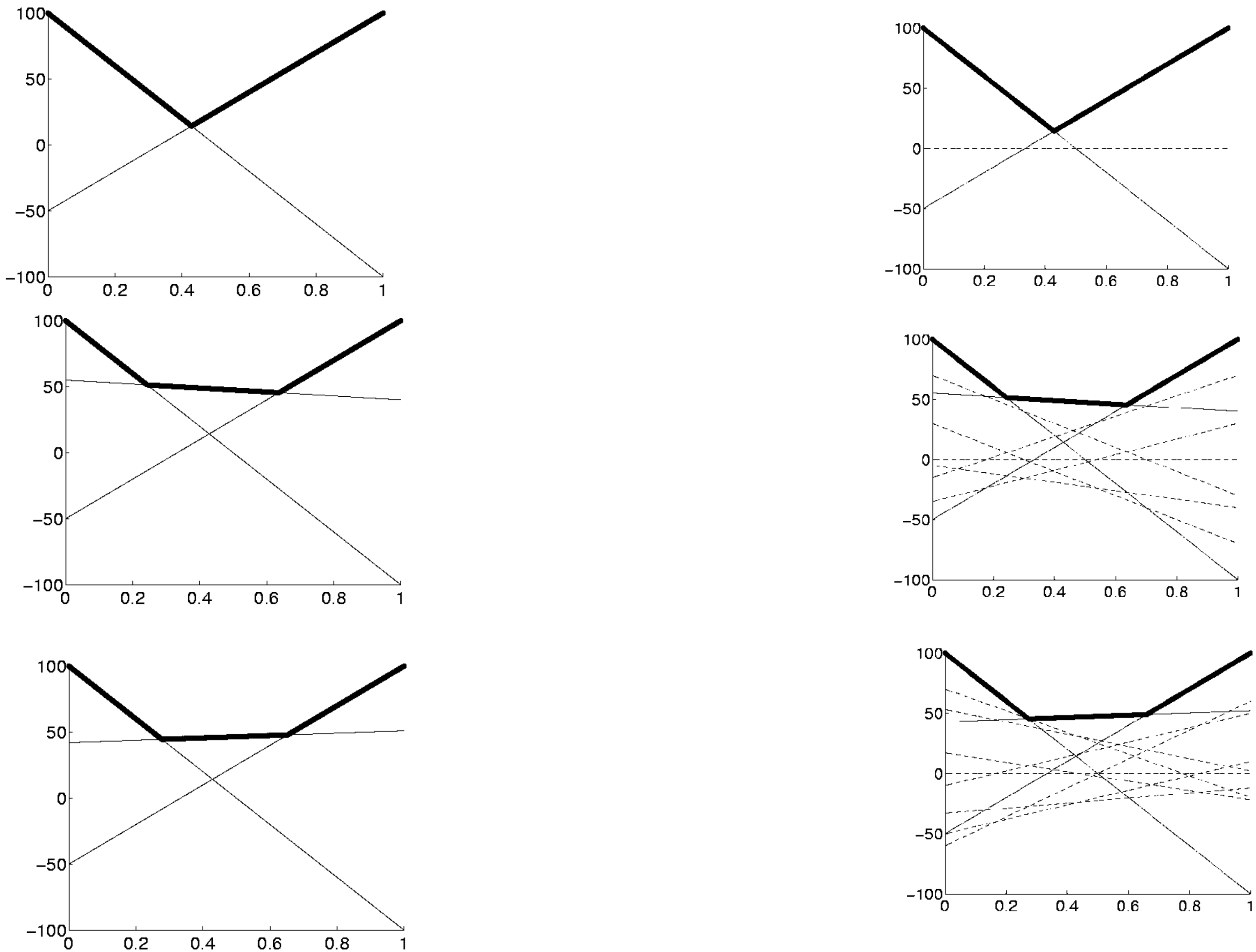
Deep Horizons and Pruning

- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20 are





Deep Horizons and Pruning



Algorithm POMDP(T):

$\Upsilon = (0, \ldots, 0)$
for $\tau = 1$ *to* T *do*
$\Upsilon' = \emptyset$
for all $(u'; v_1^k, \ldots, v_N^k)$ *in* Υ *do*
for all control actions u *do*
for all measurements z *do*
for $j = 1$ *to* N *do*

$$v_{j,u,z}^k = \sum_{i=1}^{N} v_i^k \; p(z \mid x_i) \; p(x_i \mid u, x_j)$$

endfor
endfor
endfor
endfor
for all control actions u *do*
for all $k(1), \ldots, k(M) = (1, \ldots, 1)$ *to* $(|\Upsilon|, \ldots, |\Upsilon|)$ *do*
for $i = 1$ *to* N *do*

$$v_i' = \gamma \left[r(x_i, u) + \sum_z v_{u,z,i}^{k(z)} \right]$$

endfor
add $(u; v_1', \ldots, v_N')$ *to* Υ'
endfor
endfor
optional: prune Υ'
$\Upsilon = \Upsilon'$
endfor
return Υ

Why Pruning is Essential

- Each update introduces additional linear components to V.
- Each measurement squares the number of linear components.
- Thus, an unpruned value function for T=20 includes more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.
- The pruned value functions at T=20, in comparison, contains only 12 linear components.
- The combinatorial explosion of linear components in the value function are the major reason why POMDPs are impractical for most applications.

POMDP Summary

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.
- POMDPs so far have only been applied successfully to very small state spaces with small numbers of possible observations and actions.