Lecture 11: The critical probabilty for the infinite square lattice in 2 dimensions is 1/2.

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In this lecture we prove the celebrated result, first shown by Kesten, that the critical probability of $\mathbb{L}^2$ is 1/2. The proof depends on the uniqueness of the infinite component in the supercritical phase and the fact that $\chi(p)$ is finite in the subcritical phase. These two theorems are stitched together by the self-duality of $\mathbb{L}^2$ i.e. by the fact that $\mathbb{L}^2$ and its dual $\mathbb{L}^2_{\mathrm{d}}$ are isomorphic to each other.

11.1 Preliminaries

Lemma 11.1 The "square root trick". *Given a set of increasing events A_1, $A_2, \ldots, A_m$, each having equal probability*

$$P_p(A_1) \geq 1 - \left\{1 - P_p\left(\bigcup_{i=1}^{m} A_i\right)\right\}^{\frac{1}{m}}.$$

Proof.

$$\begin{aligned} 1 - \mathrm{P}_p\left(\bigcup_{i=1}^{m} A_i\right) &= \mathrm{P}_p\left(\bigcap_{i=1}^{m} \bar{A}_i\right) \\ &\geq \prod_{i=1}^{m} \mathrm{P}_p(\bar{A}_i) \\ &= [1 - \mathrm{P}_p(A_1)]^m \end{aligned}$$

where the second step follows from the FKG inequality and the third step follows from the equiprobability of the events A_i. Taking m^{th} root on both sides, we get,

$$\left\{1 - \mathrm{P}_p(\bigcup_{i=1}^{m} A_i)\right\}^{\frac{1}{m}} \geq [1 - \mathrm{P}_p(A_1)].$$

This can be rewritten as

$$
\begin{aligned}
1-\left\{1-\mathrm{P}_p(\bigcup_{i=1}^{m} A_i)\right\}^{\frac{1}{m}} &\leq 1-[1-\mathrm{P}_p(A_1)] \\
&= \mathrm{P}_p(A_1)
\end{aligned}
$$

which proves the result. ■

11.2 Critical probability for $\mathbb{L}^2$ is 1/2.

Theorem 11.2

$$
p_c(\mathbb{L}^2)=\frac{1}{2}.
$$

Proof. We will prove this theorem in 2 parts. First, we shall show that $p_c \geq \frac{1}{2}$ by proving that $\theta(\frac{1}{2})=0$. Secondly, we shall show that $p_c \leq \frac{1}{2}$ by proving that $1-p \geq p_c > p$, for any value of p, which is violated for $(p_c >)p \geq \frac{1}{2}$.

For proving the first part, assume that $\theta(\frac{1}{2})>0$.

Define the box T(n) with $(0,0)$ as it's lower left corner and (n,n) as it's upper right corner (see Figure 1).

Define the event $A^r(n)$ to be the occurence of an open path of infinite length starting at a vertex on the right boundary of $T(n)$ which uses no other vertex of $T(n)$, as shown in Figure 1. The events $A^l(n)$, $A^t(n)$ and $A^b(n)$ are defined similarly for open paths starting at a vertex on the right, top and bottom boundary of $T(n)$ respectively.

Now, suppose that there exists a vertex in $T(n)$ which is part of an infinite cluster. Since there are only a finite number of points in $T(n)$, for finite n, the cluster will extend beyond the boundary of $T(n)$ and there would be atleast one vertex on the boundary of $T(n)$ such that there exists an open path of infinite length from it. Either this path doesn't include any other vertex from $T(n)$ or it intersects the boundary of $T(n)$ more than once, in which case, consider the path beginning at the last point of intersection of the boundary with the open path.

This shows that if a vertex in $T(n)$ is part of an infinite cluster, there is atleast one vertex on the boundary of $T(n)$ such that there exists an open path of infinite length from it which doesn't include any other vertex from $T(n)$. The converse of this statement holds trivially.

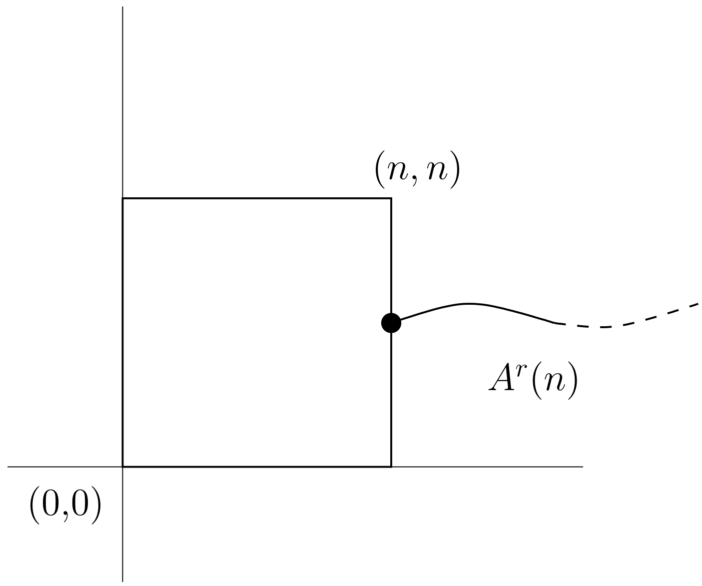

Figure 1: The box $T(n)$ and a depiction of the event $A^r(n)$.

Therefore, we can say that a vertex in $T(n)$ is part of an infinite cluster if and only if $A^l(n) \cup A^r(n) \cup A^t(n) \cup A^b(n)$ occurs.

Since we have assumed that $\theta(\frac{1}{2}) > 0$, every vertex in the first quadrant of $\mathbb{L}^2$ has a non-zero probability of being part of an open infinite cluster. Let's say the probability of $v \in \mathbb{Z}^2$ being part of an infinite open cluster is $\theta(v, \frac{1}{2}) > 0$. Clearly this event is translation invariant so we consider $\theta(0, \frac{1}{2})$.

By the FKG inequality, the probability that none of the n^2 vertices of $T(n)$ are part of on infinite open cluster with probability upperbounded by $(1 - \theta(0, \ ))^{n^2}$ which tends to 0 as $n \to \infty$. This is equivalent to saying that

$$\mathrm{P}_{\frac{1}{2}}(A^l(n) \cup A^r(n) \cup A^t(n) \cup A^b(n)) \to 1 \text{ as } n \to \infty. \tag{1}$$

Since $A^u(n)$ are identical events for $u = l, r, t, b$, $\mathrm{P}_{\frac{1}{2}}(A^u(n))$ is the same for $u = l, r, t, b$. This, along with (1) allows us to use the square root trick to say that $\mathrm{P}_{\frac{1}{2}}(A^u(n)) \to 1$ as $n \to \infty$, for $u = l, r, t, b$.

Therefore, we can choose an N such that

$$\mathrm{P}_{\frac{1}{2}}(A^u(N)) > \frac{7}{8} \text{ for } u = l, r, t, b.$$

Now, we define $T(n)_{\mathrm{d}}$, in the dual lattice, to be $\{(x_1, x_2) + (\frac{1}{2}, \frac{1}{2}) : 0 \leq x_1, x_2 \leq n\}$.

Event $A^l_{\mathrm{d}}(n)$ is defined as the occurence of an infinite closed path from a vertex on the left boundary of $T(n)_{\mathrm{d}}$, in the dual lattice, which doesn't include any other vertex of $T(n)_{\mathrm{d}}$. Events $A^r_{\mathrm{d}}(n)$, $A^t_{\mathrm{d}}(n)$ and $A^b_{\mathrm{d}}(n)$ are similarly defined for the dual lattice.

Arguing along the same lines as above, we can say that there is an N_{d} such that $P_{\frac{1}{2}}(A^u_{\mathrm{d}}(N_{\mathrm{d}})) > \frac{7}{8}$, for $u = l, r, t, b$.

Let

$$A = A^l(N^*) \cap A^r(N^*) \cap A^t_{\mathrm{d}}(N^*) \cap A^b_{\mathrm{d}}(N^*),$$

where $N^* = max(N, N_{\mathrm{d}})$. In other words we are considering the event that the left and right boundaries of $T(n)$ have vertices which lie on infinite open paths and the top and bottom boundaries of the dual box lie on infinite closed paths.

Taking complements on both sides, we get

$$\bar{A} = \overline{A^l(N^*)} \cup \overline{A^r(N^*)} \cup \overline{A^t_{\mathrm{d}}(N^*)} \cup \overline{A^b_{\mathrm{d}}(N^*)}.$$

In terms of probability,

$$\mathrm{P}_{\frac{1}{2}}(\overline{A}) \leq \mathrm{P}_{\frac{1}{2}}(\overline{A^l(N^*)}) + \mathrm{P}_{\frac{1}{2}}(\overline{A^r(N^*)}) + \mathrm{P}_{\frac{1}{2}}(\overline{A^t_{\mathrm{d}}(N^*)}) + \mathrm{P}_{\frac{1}{2}}(\overline{A^b_{\mathrm{d}}(N^*)}).$$

Note that because of the value of N^* chosen, $\mathrm{P}_{\frac{1}{2}}(A^u(N^*))$ and $\mathrm{P}_{\frac{1}{2}}(A^u_{\mathrm{d}}(N^*))$ are $> \frac{7}{8}$ and $\mathrm{P}_{\frac{1}{2}}(\overline{A^u(N^*)})$ and $\mathrm{P}_{\frac{1}{2}}(\overline{A^u_{\mathrm{d}}(N^*)})$ are $\leq \frac{1}{8}$, for $u = l, r, t, b$.

As a result, $\mathrm{P}_{\frac{1}{2}}(\overline{A}) \leq \frac{1}{2}$ and $\mathrm{P}_{\frac{1}{2}}(A) \geq \frac{1}{2}$.

Let us take a closer look at the event A. A occurs when there exist open paths of infinite length from both the left and right boundary of $T(N^*)$, which don't intersect $T(N^*)$ at any other point and closed paths of infinite length from both the top and bottom boundary of $T(N^*_{\mathrm{d}})$ in the dual lattice, which don't intersect $T(N^*_{\mathrm{d}})$ at any other point.

Since $\mathbb{L}^2$ can have atmost 1 infinite open cluster, the infinite clusters that v_l and v_r belong to must be the same i.e. v_l and v_r must be connected via an open path. Similarly, the dual lattice can have atmost 1 infinite closed cluster and hence v_t and v_b must be connected via a closed path in the dual lattice. But, as shown in Figure 2 both of these cannot happen at the same time since it would mean an open path crossing a closed path at some point which cannot happen. Thus, $P_{\frac{1}{2}}(A)$ must be 0. But we have earlier shown

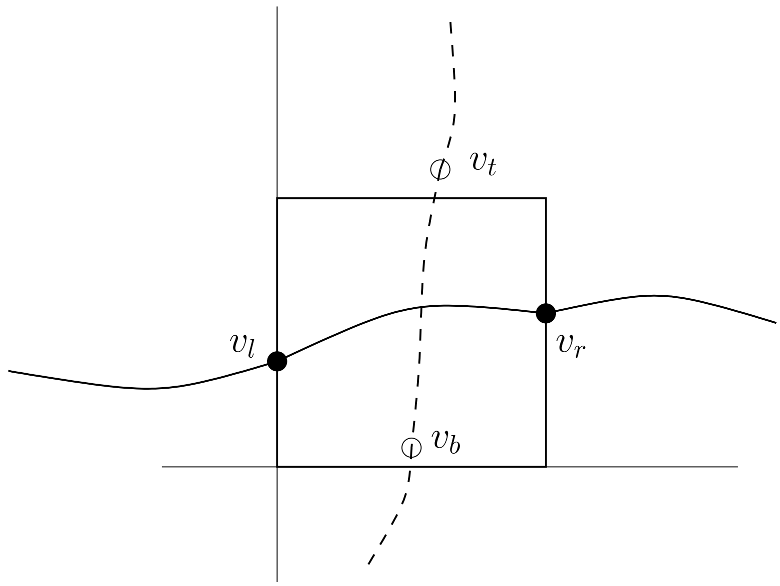

Figure 2: The infinite open cluster *and* the infinite closed cluster are unique.

that $P_{\frac{1}{2}}(A) \geq \frac{1}{2}$, under the assumption that $\theta(\frac{1}{2}) > 0$. As a result, there is a contradiction and the assumption is false. Therefore, $\theta(\frac{1}{2}) = 0$ and hence

$$p_c \geq \frac{1}{2}. \tag{2}$$

For the second part of the proof, we shall prove the following claim.

Claim 11.3 *If $p < p_c$, then there is a positive probability that the origin of the dual lattice is part of an infinite closed cluster.*

Proof of Claim 11.3. Since $p < p_c$, we are in the sub-critical phase and

$$\chi(p) = \sum_{n=1}^{\infty} \mathrm{P}_p(|C| \geq n) < \infty.$$

Given a positive integer M, we define an event A_M which occurs if there exists an open path from $(k, 0)\{\text{for some } k < 0\}$ to $(l, 0)\{\text{for some } l \geq M\}$, called π, such that all vertices of π(except the terminals) lie strictly above the horizontal axis.

Since the open paths from $(k, 0)$ to $(l, 0)$ would be of length $\geq l + |k| \geq l \geq M$, with the added constraint of lying above the horizontal axis, we get the following inequality

$$\begin{aligned} \mathrm{P}_p(A_M) &\leq \mathrm{P}_p\Big(\bigcup_{l=M}^{\infty}\{(l,0) \leftrightarrow (k,0)\}\Big) \\ &\leq \sum_{l=M}^{\infty} \mathrm{P}_p(|C_{(k,0)}| \geq l) \\ &\leq \sum_{l=M}^{\infty} \mathrm{P}_p(|C| \geq l) \end{aligned}$$

where $k < 0$ is fixed. The second inequality follows from the fact that the open cluster centered at $(k, 0)$ would be of length $\geq l$ since the open path from $(k, 0)$ to $(l, 0)$ would be of length $\geq l$. The third inequality follows from translational invariance.

Using the finiteness of $\chi(p)$ in the sub-critical phase, we can see that an infinite number of terms sum up to a finite quatity. Thus, we can find an M^* such that

$$\mathrm{P}_p(A_{M^*}) \leq \sum_{l=M^*}^{\infty} \mathrm{P}_p(|C| \geq l) \leq \frac{1}{2}.$$

This implies that $\mathrm{P}_p(\overline{A_{M^*}}) > \frac{1}{2}$.

Let $L = \{(m + \frac{1}{2}, \frac{1}{2}) : 0 \leq m < M^*\}$ and $C(L)$ be the set of vertices connected to L in the dual lattice through closed paths. If $C(L)$ is assumed to be finite, then using the same reasoning as in Lecture 3, we can say that this closed finite cluster in the dual lattice will be contained within an open circuit in the original (dual of the dual) lattice $\mathbb{L}^2$. This enclosing circuit will have the upper/lower half totally above/below the horizontal axis, intersecting the axis at 2 points, say $(k^*, 0)$ on the left, where $k^* < 0$, and $(l^*, 0)$ on the right, where $l^* > M^*$. Therefore, the upper half of the circuit can act as an open path between the 2 terminals and the event $A(M^*)$ occurs as a result. This implies that

$$\Big\{|C(L)| < \infty\Big\} \Rightarrow A_{M^*}$$

Hence

$$\mathrm{P}_p\Big(|C(L)| < \infty\Big) \leq \mathrm{P}_p(A_{M^*}) \leq \frac{1}{2}$$

and consequently

$$P_p\Big(|C(L)| = \infty\Big) \geq \frac{1}{2}.$$

Now, since $|L| = M^*$, by the Pigeonhole Principle we can say that, for some $x \in L$,

$$P_p(x \text{ is part of an infinite closed cluster}) \geq \frac{1}{2M^*}.$$

This, in turn, implies that

$$P_p(\text{the origin of } \mathbb{L}_{\mathrm{d}} \text{ is part of an infinite closed cluster}) \geq \frac{1}{2M^*} > 0,$$

because of translational invariance. This proves the claim. ∎

Since the origin is part of an infinite closed cluster with a positive probabilty, the probability that there exists an infinite closed cluster in the dual lattice must be 1. Also, the probabilty of an edge being open in the dual is still p. Flipping the open and closed edges, we see that the edges are now open with a probability $1-p$ and the infinite closed cluster will now become an infinite open cluster. This implies that we are in the super-critical phase at $1-p$. Thus the result of the claim can be interpreted as

$$p < p_c \Rightarrow p_c \leq 1 - p$$

This implies that if $\frac{1}{2} < p_c$, then $p_c \leq \frac{1}{2}$ which gives rise to a contradiction. Therefore, $p_c \leq \frac{1}{2}$. This along with (2) proves the theorem. ∎